INEQUALITIES (INEQUATIONS)

An inequality is a statement that one number is less than or greater than the other.

Symbols used

$<$ - Less than

$>$ - Greater than

$\leq$ - Less than or equal to

$\geq$ - Greater than or equal to

Interpretation of intervals

(a, b) implies $a < x < b$

$(-\infty, a)$ implies $x < a$

(a, ∞) implies $x > a$

$(-\infty, \infty)$ implies all integers

$(a, b]$ implies $a < x \leq b$

$[a, b)$ implies $a \leq x < b$

$[a, \infty)$ implies $x \geq a$

$(-\infty, a]$ implies $x \leq a$

Absolute value of an inequality.

(i) If $|P| = b$, then $P = b$ and $P = -b$

(ii) If $|P| \leq b$, then $-b < P < b$

(iii) If $|P| > b$, then $P < -b$ and $P > b$

Solving inequalities

The following rules are observed when solving inequalities.

(i) We may add any integer, Positive or negative, to each side of the inequality.

Eg If $3 < 7$ then,

$3 + 2 < 7 + 2$

If $5 < 12$ then,

$5 + (-2) < 12 + (-2)$

(ii) We may multiply or divide each side of the inequality by a positive integer.

E.g If $4 < 7$ then,

$4 \times 2 < 7 \times 2$

If $11 > 6$ then,

$\frac{11}{3} > \frac{6}{3}$

(iii) If each side of the inequality is divided or multiplied by a negative integer, the inequality sign reverses i.e $>$ becomes $<$ and $<$ becomes $>$

E.g If $3 < 7$ then,

$3 \times -3 > 7 \times -3$

If $5 < 9$ then,

$\frac{5}{-3} > \frac{9}{-3}$

Examples

① Solve the following inequalities:

(i) $8 - 2x \leq 3$.

(ii) $|2x - 3| \leq 7$

(iii) $|3x + 1| > 8$

Soln

Soln

(i) $8 - 2x \leq 3$

$-2x \leq 3 - 8$

$-2x \leq -5$

$\therefore x \geq 2.5$

(ii) $-7 < 2x - 3 < 7$

$-4 < 2x < 10$

$\therefore -2 < x < 5$

(iii) $3x + 1 > 8$ and $3x + 1 < -8$

For $3x + 1 > 8$ For $3x + 1 < -8$

$3x > 7$ $3x < -9$

$x > 7/3$ $x < -3$

② Find the range of values of x for which;

(i) $\frac{(x-1)(x-3)}{(x+1)(x-2)} > 0$

(ii) $\frac{3-x}{x+2} < 4$

(iii) $|2x-3| > |x+3|$

Soln

(i) let $\frac{(x-1)(x-3)}{(x+1)(x-2)} = y$; $y > 0$ ie +ve

Critical values of x,

x is $\{-1, 1, 2, 3\}$

Table of sign inspection.

	$x<-1$	$-1<x<1$	$1<x<2$	$2<x<3$	$x>3$
$x-1$	−	−	+	+	+
$x-3$	−	−	−	−	+
$x+1$	−	+	+	+	+
$x-2$	−	−	−	+	+
Sign on y	+	−	+	−	+

$\therefore x < -1$, $1 < x < 2$ and $x > 3$

(ii) $\frac{3-x}{x+2} - 4 < 0$

$$\frac{3-x-4(x+2)}{x+2} < 0$$

$$\frac{-5-5x}{x+2} < 0$$

let $\frac{-5-5x}{x+2} = y$

$\Rightarrow y < 0$ i.e $-ve$

Critical values of x

$x = \{-2, -1\}$

Table of sign inspection

	$x<-2$	$-2<x<-1$	$x>-1$
$-5-5x$	+	+	−
$x+2$	−	+	+
Sign on y	−	+	−

$\therefore$ $x<-2$ and $x>-1$

(iii) $(2x-3)^2 > (x+3)^2$

$4x^2-12x+9 > x^2+6x+9$

$$3x^2-18x > 0$$

$$3x(x-6) > 0$$

$$x(x-6) > 0$$

let $x(x-6) = y$

$\Rightarrow y > 0$ i.e $+ve$

Critical values of x

x is $\{0, 6\}$.

Table of sign inspection

	$x<0$	$0<x<6$	$x>6$
x	−	+	+
$x-6$	−	−	+
sign on y	+	−	+

$\therefore$ $x<0$ and $x>6$

③ Given that $(x^2-x+1)y = 2x$, Find the range of the values of y for which the equation has real roots.

Soln

$yx^2 - yx - 2x + y = 0$

$yx^2 - (y+2)x + y = 0$

$a = y$, $b = -(y+2)$ $c = y$

for real roots, $b^2 \geq 4ac$

$\Rightarrow [-(y+2)]^2 - 4(y)(y) \geq 0$

$y^2+4y+4 - 4y^2 \geq 0$

$(3y+2)(y-2) \leq 0$

Let $(3y+2)(y-2) = f(y)$

$\Rightarrow f(y) \leq 0$ ie $-ve$

critical values of y

$y = \{-2/3, 2\}$

Table of sign inspection

	$y<-2/3$	$-\frac{2}{3}\leq y\leq 2$	$y>2$
$3y+2$	−	+	+
$y-2$	−	−	+
sign on f(y)	+	−	+

$\therefore$ The range is $-2/3 \leq y \leq 2$

④ Represent the inequalities $y > x-5$ and $0 < y < 6/x$ on the same axes and shade the unwanted region.

Soln

For $y > x-5$

The boundary line is $y = x - 5$

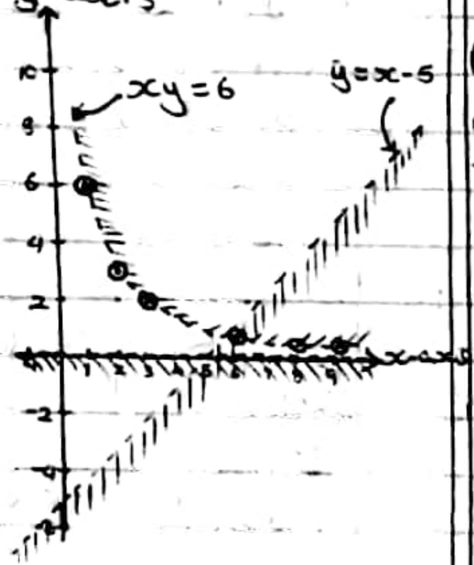
x	0	5
y	-5	0
(x, y)	$(0, -5)$	$(5, 0)$

For $0 < y < 6/x$

Boundary lines are $xy = 6$ and $y = 0$.

considering $xy = 6$

x	1	2	3	6	8	10
y	6	3	2	1	$3/4$	$3/5$



Exercise

① Solve the inequality;

$(0{\cdot}8)^x > 4{\cdot}0$

② For what range of values of x do the following inequalities hold?

(i) $\dfrac{x-1}{x-2} > \dfrac{x-2}{x+3}$

(ii) $\dfrac{2x^2-7x-4}{3x^2-14x+11} > 2$

(iii) $\dfrac{(x-1)(x+3)}{(x+1)(x-2)} < 0$

(iv) $\dfrac{3x^2-1}{x+2} \geq 2$

(v) $\dfrac{3x}{x-8} < \dfrac{2x-1}{x-5}$

(vi) $x > \dfrac{2}{x+1}$

(vii) $2(x+1) \leq x-2$

(viii) $|x-1| \geq |2x-4|$

(ix) $x+6 > |2x+3|$

$(-3 < x < 3)$

③ Find the range of the possible values of y for real values of x in each of the equations below:

(i) $y = \dfrac{3-2x}{4+x^2}$

(ii) $y = \dfrac{3x+3}{x(3-x)}$

(iii) $y = \dfrac{3x^2-1}{x+2}$

(iv) $y = \dfrac{4(x-3)}{x(x+2)}$

④ Given that $y = \dfrac{3x-6}{x^2+6x}$, find the range of values within which y does not lie

SURDS

A surd is an expression containing at least a square root of a prime number and as such can not be evaluated exactly.

All surds are irrational numbers i.e they can not be expressed in the form a/b where a and b are positive integers.

Examples of surds include $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $2\sqrt{3}$, $3\sqrt{5}$, $4\sqrt{7}$, etc

ALGEBRA OF SURDS

(I) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(II) $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$

(III) $\sqrt{a}/\sqrt{b} = \sqrt{a/b}$

(IV) $(a \pm \sqrt{b})^2 = a^2 + b \pm 2a\sqrt{b}$

(V) $(\sqrt{a} \pm \sqrt{b})^2 = a + b \pm 2\sqrt{ab}$

NB:

When surds appear in the denominator, they are always removed. The process of removing surds from the denominator is called rationalising the denominator.

To rationalise the fraction, we multiply its numerator and denominator by the conjugate of its denominator.

The conjugate of $p+q$ is $p-q$ where p or q or both is/are in surd form e.g the conjugate of $\sqrt{a}+b$ is $\sqrt{a}-b$ and the conjugate of $-\sqrt{a}+b$ is $-\sqrt{a}-b$.

Examples

① Simplify the following,

(I) $\sqrt{48}$ (II) $\sqrt{50} + 2\sqrt{18} - \sqrt{32}$

② Express the following in the form $a + b\sqrt{c}$

(I) $\dfrac{2+\sqrt{3}}{1-\sqrt{3}}$ (II) $\dfrac{(2+\sqrt{5})(3+\sqrt{5})(\sqrt{5}-2)}{(\sqrt{5}-1)(1+\sqrt{5})}$

③ Find the square root of $6 + 2\sqrt{5}$.

Soln

Let the square root be $\pm(\sqrt{x_1} + \sqrt{x_2})$ $\pm(1+\sqrt{5})$

$x_1 + x_2 + 2\sqrt{x_1x_2} = 6 + 2\sqrt{5}$

$x_1 + x_2 = 6$ and $x_1x_2 = 5$

Exercise

① Express the following as fractions with rational denominators:

(a) $\frac{-2+\sqrt{3}}{-2-\sqrt{3}}$ (b) $\frac{\sqrt{3}-2}{2\sqrt{3}+3}$

(c) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ (d) $\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}$

(e) $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ (f) $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$

② Simplify the following fractions

(a) $\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{7}}$ (b) $\frac{1}{3\sqrt{5}-\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{3}}$

③ Show that $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\frac{3+\sqrt{6}+\sqrt{15}}{6}$

④ Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$.

⑤ Given that $t=\frac{1}{2}(\sqrt{5}+1)$, show that $t^2=1+t$.

⑥ Given that $a=b\sqrt{2}$, show that

$$\frac{a^3-2a^2b+b^3}{ab(a+3b)}=\frac{2+11\sqrt{2}}{14}$$

⑦ Given that $\sqrt{52-30\sqrt{3}}=p+q\sqrt{3}$, find the values of p and q:

⑧ Given that $\sqrt{3}=1.732$ and $\sqrt{2}=1.414$, by rationalising the denominator, evaluate $\frac{1}{\sqrt{3}-\sqrt{2}}$ to 3 significant figures

⑨ Find the square root of $8-2\sqrt{15}$.

$[\pm(\sqrt{5}-\sqrt{3})]$

LOGARITHMS AND INDICES

INDICES

An index is the power to which a given number is raised e.g for y^x, x is the index and y is the base

Laws of indices

For any positive integers m and n,

(i) $x^n \times x^m = x^{m+n}$

(ii) $x^n \div x^m = x^{n-m}$

(iii) $(x^n)^m = x^{mn}$

(iv) $x^{m/n} = \sqrt[n]{x^m}$

(v) $x^{-n} = \left(\frac{1}{x}\right)^n$, provided $x \neq 0$

NB:

$x^0 = 1$ provided $x \neq 0$

proof

$$x^0 = x^{n-n}$$
$$= x^n \div x^n$$
$$= \frac{x^n}{x^n}$$
$$= 1$$
$$\therefore x^0 = 1$$

Examples

① Simplify each of the following expressions.

(a) $4^{3/2}$ (b) $\left(\frac{2}{3}\right)^{-3}$ (c) $\left(\frac{1}{8}\right)^{-4/3}$ (d) $\frac{4^{-1} \times 9^{\frac{1}{2}}}{8^{-2}}$

Soln

(a) $4^{3/2} = (2^2)^{3/2}$

$= 2^{(2 \times \frac{3}{2})}$

$= 2^3$

(b) $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3$

$= \frac{3^3}{2^3}$ or $3^3 \times 2^{-3}$

(c) $\left(\frac{1}{8}\right)^{-4/3} = (8)^{4/3}$

$= (2^3)^{4/3}$

$= 2^{3\times 4/3}$

$= 2^4$

(d) $\frac{4^{-1}\times 9^{1/2}}{8^{-2}} = \frac{(2^2)^{-1}\times(3^2)^{\frac{1}{2}}}{(2^3)^{-2}}$

$= \frac{2^{-2}\times 3}{2^{-6}}$

$= 2^{-2-(-6)}\times 3$

$= 2^4\times 3$

② Solve each of the equations below:

(a) $x^{1/3} = 3$ (b) $x^{4/3} = 81$

(c) $2x^{3/4} = x^{1/2}$ (d) $x^{1/3} - 3 = 28x^{-1/3}$

Soln

(a) $x^{1/3} = 3$

$(x^{1/3})^3 = 3^3$

$\therefore x = 27$

(b) $x^{4/3} = 3^4$

$(x^{4/3})^3 = (3^4)^3$

$x^4 = 3^{12}$

$= (3^3)^4$

$\therefore x = 27$

(c) $2x^{3/4} = x^{1/2}$

$\frac{x^{3/4}}{x^{1/2}} = \frac{1}{2}$

$x^{3/4-\frac{1}{2}} = 2^{-1}$

$x^{1/4} = 2^{-1}$

$(x^{1/4})^4 = (2^{-1})^4$

$\therefore x = (\frac{1}{2})^4$

$= 1/32$

(d) $x^{1/3} - 3 = 28x^{-1/3}$

$x^{1/3}.x^{1/3} - 3x^{1/3} = 28x^{-1/3}.x^{1/3}$

$x^{2/3} - 3x^{1/3} = 28$

let $x^{1/3} = a$

$\Rightarrow a^2 - 3a - 28 = 0$

Either $a = -4$ or $a = 7$

$\therefore$ Either $x = -64$ or $x = 343$

---

③ Solve the equation $2\sqrt{x-1} - \sqrt{x+4} = 1$

Soln

$2\sqrt{x-1} = 1 + \sqrt{x+4}$

$4(x-1) = 1 + 2\sqrt{x+4} + x + 4$

$4x - 4 = 1 + 2\sqrt{x+4} + x + 4$

$3x - 9 = 2\sqrt{x+4}$

$9x^2 - 58x + 65 = 0$

$x = 5$ or 1.44444

Test to eliminate the extraneous root which results from double squaring.

$\therefore x = 5$

---

④ Simplify; $\frac{(1+x)^{1/2} - \frac{1}{2}x(1+x)^{-1/2}}{1+x}$

Soln

$$\frac{(1+x)^{\frac{1}{2}} - \frac{1}{2}x(1+x)^{-1/2}}{1+x} = (1+x)^{-1/2}\left[\frac{(1+x) - \frac{1}{2}x}{1+x}\right]$$

$$= \frac{(1+x)^{\frac{1}{2}}[2(1+x) - x]}{2(1+x)}$$

$$= \frac{2+x}{2(1+x)^{3/2}}$$

OR, Split the fraction

Alternatively;

$$\frac{(1+x)^{\frac{1}{2}} - \frac{1}{2}x(1+x)^{-1/2}}{(1+x)} = \left[\frac{(1+x)^{\frac{1}{2}} - \frac{1}{2}x(1+x)^{-\frac{1}{2}}}{1+x}\right]\left(\frac{(1+x)^{-\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}}\right)$$

$$= \frac{\frac{1}{2}x^{\frac{1}{2}}(1+x)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{x}{(1+x)^{\frac{1}{2}}}\right)}{1+x}$$

$$= \frac{2+x}{2(1+x)^{3/2}}$$

Exercise

① Simplify the following expressions

(a) $(x-2)^{5/2} + 2(x-2)^{3/2}$ (b) $\dfrac{8^{n+2} \times 4^{2n-1}}{2^n \times 4^{n/2}}$

(c) $\dfrac{3(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n}$ (d) $\dfrac{\frac{1}{2}x^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}}{x}$

(4) $\left(\dfrac{-1}{2x^{3/2}(1+x)^{1/2}} \right)$

② Express (i) $4^{(1-\frac{n}{2})} \times 2^{n+3} \times 16^{-1/2}$ in powers of 2

(ii) $9^{(1+\frac{n}{2})} \times 3^{n-3} \times 81^{-1/2}$ in the form 3^n

③ Solve each of the following equations

(a) $5^x = 25^{(2x-5)}$ (b) $8^x = 4^{(1-x)}$

(c) $\left(\frac{1}{4}\right)^x \times 2^{x+1} = \frac{1}{8}$ (d) $5^x = 125\sqrt{5}$

(e) $4^{x-1} = 3^{x+1}$ (f) $6^x = \left(\frac{2}{3}\right)^{(x-\frac{1}{2})}$

(g) $x^3 - 8 - 9x^{3/2}$ (h) $6x^{1/3} + 5 + x^{-1/3} = 0$

(i) $27(3^x) - 3^{2x} = 6$ (j) $2^{2(x-1)} - 3(2^x) + 8 = 0$

(k) $3^{(2x+1)} + 26(3^x) = 9$ (l) $3^{(2x+1)} - 3^{(x+1)} - 3^x + 1 = 0$

(m) $2^{(2x-1)} + \frac{3}{2} = 2^{(x+1)}$ (n) $x^4 - 4x^2 + 3 = 0$

(o) $2^{(2x+3)} + 1 = 9(2^x)$ (p) $3(3^{2x}) + 2(3^x) - 2 = 0$

④ Solve the following equations and justify your answers

(a) $2\sqrt{x} + \sqrt{2x+1} = 7$

(b) $\sqrt{3x+4} - \sqrt{x+7} = 3$

(c) $\sqrt{2-x} + \sqrt{x+3} = 3$

(d) $\sqrt{2x+3} - \sqrt{x+1} = \sqrt{x-2}$

$(x = 3)$

LOGARITHMS

The logarithm of a positive quantity N to a given base a is the index (power) to which a must be raised to make it equal to N. i.e If $a^x = N$, then $x = \log_a N$.

Basic laws of logarithms

(I) $\log_c c = 1$

(II) $\log_c 1 = 0$

(III) $a^{\log_a x} = x$

(IV) $\log_c a^n = n \log_c a$

(V) $\log_c ab = \log_c a + \log_c b$

(VI) $\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b$

(VII) $\log_a b = \frac{\log_c b}{\log_c a}$

Proofs

(I) $\log_c c = 1$

Let $\log_c c = a$

$\Rightarrow c^a = c$

$c^a = c^1$

$a = 1$

$\therefore \log_c c = 1$

(II) $\log_c 1 = 0$

Let $\log_c 1 = a$

$\Rightarrow c^a = 1$

$c^a = c^0$

$a = 0$

$\therefore \log_c 1 = 0$

(iii) $a^{\log_a x} = x$

Let $a^n = x$

$\Rightarrow \log_a x = n$

$\therefore a^{\log_a x} = x$

(iv) $\log_c a^n = n \log_c a$

Let $\log_c a = x$

$\Rightarrow c^x = a$

$c^{nx} = a^n$

$\log_c c^{nx} = \log_c a^n$

$nx \log_c c = \log_c a^n$

$\therefore \log_c a^n = n \log_c a$

(v) $\log_c ab = \log_c a + \log_c b$

Let $x = \log_c a$ and $y = \log_c b$.

$\Rightarrow c^x = a$ and $c^y = b$

$ab = c^x \cdot c^y$

$= c^{x+y}$

$= c^{(\log_c a + \log_c b)}$

$\log_c ab = \log_c c^{(\log_c a + \log_c b)}$

$\therefore \log_c ab = \log_c a + \log_c b$

(vi) $\log_c (a/b) = \log_c a - \log_c b$

Let $x = \log_c a$ and $y = \log_c b$

$\Rightarrow c^x = a$ and $c^y = b$

$\frac{a}{b} = \frac{c^x}{c^y}$

$= c^{x-y}$

$= c^{(\log_c a - \log_c b)}$

$\log_c \left(\frac{a}{b}\right) = \log_c c^{(\log_c a - \log_c b)}$

$\log_c (a/b) = (\log_c a - \log_c b) \log_c c$

$\therefore \log_c (a/b) = \log_c a - \log_c b$

(vii) $\log_a b = \dfrac{\log_c b}{\log_c a}$

Let $\log_a b = x$

$\Rightarrow a^x = b$

$\log_c a^x = \log_c b$

$x \log_c a = \log_c b$

$x = \dfrac{\log_c b}{\log_c a}$

$\therefore \log_a b = \dfrac{\log_c b}{\log_c a}$

Examples

① Solve the equations below.

(i) $\log_5 x + \log_x 5 = 2.5$

(ii) $\log_4 (6-x) = \log_2 x$

Soln

(i) $\log_5 x + \log_x 5 = 2.5$

$\log_5 x + \dfrac{\log_5 5}{\log_5 x} = 2.5$

Let $\log_5 x = a$

$\Rightarrow a + \dfrac{1}{a} = 2.5$

$2a^2 + 2 = 5a$

$2a^2 - 5a + 2 = 0$ $\quad$ 4; -4 -1

$2a^2 - 4a - a + 2 = 0$

$2a(a-2) - (a-2) = 0$

$(2a-1)(a-2) = 0$

either $2a - 1 = 0$; $a = \frac{1}{2}$

or $a - 2 = 0$; $a = 2$

When $a = 2$, $\log_5 x = 2$

$\Rightarrow x = 5^2 = 25$

When $a = \frac{1}{2}$, $\log_5 x = \frac{1}{2}$

$\Rightarrow x = \sqrt{5}$

$\therefore$ Either $x = 25$ or $x = \sqrt{5}$

(II) $\log_4(6-x) = \log_2 x$

$$\Rightarrow \frac{\log_2(6-x)}{\log_2 4} = \log_2 x$$

$$\log_2(6-x) = \log_2 x^2$$

$$6 - x = x^2$$

$$x^2 + x - 6 = 0 \qquad \begin{array}{c} -6 \\ 3 \quad -2 \end{array}$$

$$x^2 + 3x - 2x - 6 = 0$$

$$x(x+3) + 2(x-3) = 0$$

$$(x-2)(x+3) = 0$$

Either $x + 3 = 0$; $x = -3$ (discard)

or $x - 2 = 0$; $x = 2$

$\therefore\ x = 2$

---

② Show that $\log_6 x = \dfrac{\log_3 x}{1 + \log_3 2}$ hence evaluate $\log_6 4$ given that $\log_3 2 = 0.631$

Soln

$$\log_6 x = \frac{\log_3 x}{\log_3 6}$$

$$= \frac{\log_3 x}{\log_3(2\times 3)}$$

$$= \frac{\log_3 x}{\log_3 3 + \log_3 2}$$

$$= \frac{\log_3 x}{1 + \log_3 2}$$

Hence $\log_6 4 = \dfrac{\log_3 2^2}{1 + \log_3 2}$

$$= \frac{2\log_3 2}{1 + \log_3 2}$$

$$= \frac{2(0.631)}{1 + 0.631}$$

$$= 0.774$$

---

③ Solve the equations $5\log_x y = 1$ and $xy = 64$ simultaneously

Scanned by CamScanner

Soln

$5\log_x y = 1$ ---------- (i)

$xy = 64$ ---------- (ii)

from (i),

$x = y^5$

Substitute for x in (ii)

$\Rightarrow y^6 = 64$

$y^6 = 2^6$

$y = 2$

$\Rightarrow x = 2^5 = 32$

$\therefore x = 32$ and $y = 2$

---

④ Prove that $\log_c(a+b)^2 = 2\log_c a + \log_c\left[1 + \frac{2b}{a} + \frac{b^2}{a^2}\right]$

Soln

LHS $= \log_c(a+b)^2$

$= \log_c(a^2 + 2ab + b^2)$

$= \log_c a^2\left[1 + \frac{2b}{a} + \frac{b^2}{a^2}\right]$

$= \log_c a^2 + \log_c\left[1 + \frac{2b}{a} + \frac{b^2}{a^2}\right]$

$= 2\log_c a + \log_c\left[1 + \frac{2b}{a} + \frac{b^2}{a^2}\right]$

$\therefore \log_c(a+b)^2 = 2\log_c a + \log_c\left[1 + \frac{2b}{a} + \frac{b^2}{a^2}\right]$

---

⑤ If $\log_a n = x$ and $\log_c n = y$ where $n \neq 1$, prove that $\frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$

Soln

LHS $= \frac{x-y}{x+y}$

$= \frac{\log_a n - \log_c n}{\log_a n + \log_c n}$

$= \frac{\left(\frac{\log_b n}{\log_b a}\right) - \left(\frac{\log_b n}{\log_b c}\right)}{\left(\frac{\log_b n}{\log_b a}\right) + \left(\frac{\log_b n}{\log_b c}\right)}$

Scanned by CamScanner

$$= \frac{\log_b n}{\log_b n}\left[\frac{\frac{1}{\log_b a} - \frac{1}{\log_b c}}{\frac{1}{\log_b a} + \frac{1}{\log_b c}}\right]$$

$$= \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$$

$$\therefore \frac{x-y}{x+y} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$$

---

(6) Given that $\log_b c = a$, $\log_c a = b$ and $\log_a b = c$, prove that $abc = 1$

Soln

$a = \log_b c \;;\; b^a = c$ - - - - - - - - - (I)

$b = \log_c a \;;\; c^b = a$ - - - - - - - - - (II)

$c = \log_a b \;;\; a^c = b$ - - - - - - - - - (III)

Substitute for c in (II)

$$(b^a)^b = a$$

Substitute for a in (III)

$$(b^{ab})^c = b$$

$$b^{abc} = b^1$$

$$\therefore abc = 1$$

Alternatively LHS $= abc$
$= \log_b c \cdot \log_c a \cdot \log_a b$

---

(7) Given that $x = \log_a bc$, $y = \log_b ac$ and $z = \log_c ab$, prove that:

$$x + y + z = xyz - 2$$

Soln

$x = \log_a bc \;;\; a^x = bc$ - - - - - - - - (1)

$y = \log_b ac \;;\; b^y = ac$ - - - - - - - - (II)

$z = \log_c ab \;;\; c^z = ab$ - - - - - - - - (III)

From (1), $a = b^{1/x} c^{1/x}$

Substitute for a in (II)

$$b^y = b^{1/x} c^{1/x} c$$

$$\Rightarrow c = b^{\frac{xy-1}{x+1}} \text{ - - - - - - - - - } (*)$$

Substitute for a in (III)

$$c^z = b^{1/x} c^{1/x} b$$

$\Rightarrow \; C = b^{\frac{1+x}{2x-1}}$ - - - - - - - - - - - - - - - (**)

On equating (*) and (**),

$$\frac{1+x}{2x-1} = \frac{xy-1}{x+1}$$

$$1 + 2x + x^2 = x^2yz - xz - xy + 1$$

$$\therefore \; x + y + z = xyz - 2$$

Exercise

① Without using tables or calculator, evaluate $\frac{\log 81}{\log 729}$

② Simplify the following logarithms.

(i) $1 + \log_{10}\left(\frac{4}{x^4}\right)^{-\frac{1}{2}} - 2\log_{10} x$

(ii) $\log\sqrt{x^2-1} + \frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$

③ Solve the equations below

(i) $\log_n 4 + \log_4 n^2 = 3$

(ii) $\log_4 14x = 2\log_x 4$
($x = 4$)

(iii) $3\log_2 P - 6\log_P 2 + 7 = 0$

(iv) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{16}$

(v) $\log_{9x} 64 = \log_x 4$ (vi) $2^x \cdot 3^{2x} = 5$

④ Solve the following pairs of simultaneous equations.

(i) $\log_{10} x - \log_{10} y = \log_{10} 2\cdot 5$ and $\log_{10} x + \log_{10} y = 1$

(ii) $\log_b a + 2\log_a b = 3$ and $\log_9 a + \log_9 b = 3$ where $a \neq b$

(iii) $6\log_8 x + 6\log_{27} y = 7$ and $4\log_2 x - 4\log_3 y = 9$

⑤ Given that $\log_b a = x$, show that $b = a^{1/x}$ hence prove that $\log_a b = \frac{1}{\log_b a}$

⑥ Given that $\log_3 x = p$ and $\log_{18} x = q$, show that $\log_6 3 = \frac{q}{p-q}$.

⑦ Given that $a = \log_5 35$ and $b = \log_9 35$, show that $\log_5 21 = \frac{1}{2b}(2ab - 2b + a)$

⑧ Given that $2\log_8 N = p$, $\log_2 2N = q$ and $q - p = 4$, Find the value of N.

($N = 512$)

⑨ Given that $P = \log_a(a^3 y^{-2})$ and $q = \log_a(ay^2)$, find $P + q$.

⑨ Show that $\log_8 x = \frac{2}{3}\log_4 x$ hence without using tables or calculator evaluate $\log_8 6$ if $\log_4 3 = 0.7925$.

(0.862)

⑩ If $\log_a(1 + \frac{1}{24}) = n$, $\log_a(1 + \frac{1}{8}) = L$ and $\log_a(1 + \frac{1}{25}) = m$ show that;

$L - m - n = \log_a(1 + \frac{1}{80})$

⑪ Given that $\log(p - q + 1) = 0$ and $\log(pq) + 1 = 0$, show that $p - q = \frac{1}{10}\sqrt{10}$

⑫ The variable x satisfies the equation $3^x \cdot 4^{2x+1} = 6^{x+2}$. By taking logarithms on both sides, show that $x = \frac{\log 9}{\log 8}$

⑬

THE RATIO THEOREM

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = \lambda$, then;

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots = \frac{la + mc + ne + \dots}{lb + md + nf + \dots},$$

where $l, m, n, \dots$ are non zero constants.

Proof

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{la + mc + ne}{lb + md + nf}$$

let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

$\Rightarrow \frac{a}{b} = k$; $a = kb$

Similarly, $c = kd$ and $e = kf$

$$\Rightarrow RHS = \frac{lbk + mdk + nfk}{lb + md + nf}$$

$$= k$$

$\therefore$ LHS = RHS hence proved.

Examples

① If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a-c}{a+c} = \frac{b-d}{b+d}$

Soln

let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow \frac{a}{b} = k$; $a = bk$

Similarly, $c = dk$

$$\Rightarrow \frac{a-c}{a+c} = \frac{bk - dk}{bk + dk}$$

$$= \frac{k(b-d)}{k(b+d)}$$

$$= \frac{b-d}{b+d}$$

$$\therefore \frac{a-c}{a+c} = \frac{b-d}{b+d}$$

② If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a+2c}{b+2d} = \frac{3a+c}{3b+d}$

Soln

Let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow \frac{a}{b} = k, \quad a = bk$

Similarly $c = dk$

$\Rightarrow \frac{a+2c}{3a+c} = \frac{kb+2kd}{3kb+kd}$

$= \frac{k(b+2d)}{k(3b+d)}$

$= \frac{b+2d}{3b+d}$

On rearranging,

$\frac{a+2c}{b+2d} = \frac{3a+c}{3b+d}$

③ If $\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)} = \frac{a}{b}$,

Prove that:

$\frac{b+a}{b-a} = -\tan\theta\cot\phi$

Soln

$\frac{b+a}{b-a} = \frac{\sin(\theta-\phi)+\sin(\theta+\phi)}{\sin(\theta-\phi)-\sin(\theta+\phi)}$

$= \frac{2\sin\theta\cos(-\phi)}{2\cos\theta\sin(-\theta)}$

$= -\tan\theta\cot\phi$

④ If $\tan\theta = \frac{a}{b}$, show that

$\frac{a}{a+b} = \frac{\sin\theta}{\sqrt{2}\sin(\theta+\pi/4)}$

Soln

$\frac{a}{b} = \frac{\sin\theta}{\cos\theta}$

$\frac{a}{a+b} = \frac{\sin\theta}{\sqrt{2}\sin(\theta+\pi/4)}$

$= \frac{\sin\theta}{\sqrt{2}\sin(\theta+\frac{\pi}{4})}$

⑤ Solve the equations

$\frac{y+z}{5} = \frac{z+x}{8} = \frac{x+y}{9}$

and $6(x+y+z) = 11$

Simultaneously.

Soln

Let $\frac{y+z}{5} = \frac{z+x}{8} = \frac{x+y}{9} = k$

$\frac{y+z+z+x+x+y}{5+8+9} = k$

$\frac{2(x+y+z)}{11} = k$

$\Rightarrow k = 1/6$

$\therefore x = 1, \ y = \frac{1}{2}$ and $z = \frac{1}{3}$

Alternatively; Equate half equations.

Exercise

① Solve the equations below simultaneously

(i) $\frac{4x-3y}{4} = \frac{2y-x}{3} = \frac{z-4y}{2}$ and $3x+3y+z = 3$

(ii) $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}$ and $4x+2y+5z = 30$

(iii) $x+2y = \frac{y+2z}{4} = \frac{2x+2}{5}$ and $x+y+z = 2$

(iv) $x = \frac{x+y}{3} = \frac{x-y+z}{2}$ and $x^2+y^2+z^2+x+2y+4z = 6$

(v) $\frac{x-y}{4} = \frac{z-y}{3} = 2z-x$ and $x+3y+2z = 4$

② Given that $\tan\theta = \frac{a}{b}$, Find the expression for $\frac{a^2}{a+b}$

POLYNOMIALS

A polynomial is an expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_nx^0$ where $a_0, a_1, a_2, \dots, a_{n-1}$ and a_n are real numbers and n is a positive integer.

The positive integer n is the degree of the polynomial e.g $x^4 + 2x^3 + 7x^2 + x + 3$ is a polynomial of degree 4.

THE REMAINDER THEOREM

If $Q(x)$ is the quotient and $R(x)$ is the remainder when a polynomial $f(x)$ is divided by $g(x)$ then $f(x) = g(x)Q(x) + R(x)$

NB:

When a polynomial is divided by a function of degree n, the remainder is of degree $n-1$ e.g. If a polynomial is divided by a quadratic expression the remainder is linear and if a polynomial is divided by a cubic expression the remainder is a quadratic expression.

When a polynomial $f(x)$ is divided by the expression $(x-\alpha)$, the remainder is $f(\alpha)$. If $f(\alpha)=0$, then $(x-\alpha)$ is a factor of $f(x)$.

Examples

① Divide $x^3 - 3x^2 + 6x + 5$ by $x-2$

Soln

Using long division

$$
\begin{array}{r|l}
 & x^2 - x + 4 \\
\hline
x-2 & x^3 - 3x^2 + 6x + 5 \\
- & x^3 - 2x^2 + 0 + 0 \\
\hline
 & -x^2 + 0 + 0 \\
- & x^2 + 2x + 0 \\
\hline
 & 4x + 5 \\
 & 4x - 8 \\
\hline
 & +13
\end{array}
$$

$$\therefore \frac{x^3 - 3x^2 + 6x + 5}{x-2} = x^2 - x + 4 + \frac{13}{x-2}$$

Using detached coefficient method
(Synthetic division)
The divisor is $x-2$
$\Rightarrow x-2=0;\ x=2$

Root \ Term	x^3	x^2	x^1	x^0
2	1	-3	6	5
+	0	2	-2	8
	1	-1	4	13

coefficients of the quotient — Remainder

$$\therefore \frac{x^3-3x^2+6x+5}{x-2} = x^2-x+4 + \frac{13}{x-2}$$

---

② When a polynomial $P(x)$ is divided by $x-1$, the remainder is 5. When $P(x)$ is divided by $x-2$, the remainder is 7. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

Soln

Let the remainder be $ax+b$.

$\Rightarrow P(x) = (x-1)(x-2)Q(x) + (ax+b)$

$P(1) = a+b = 5$ - - - - - - - - - (I)

$P(2) = 2a+b = 7$ - - - - - - - - - (II)

From (I) and (II), $a = 2$ and $b = 3$

$\therefore$ The remainder is $2x+3$

---

③ Factorise x^3-2x^2-5x+6 completely
[$(x-3)(x-1)(x+2)$]

---

④ A polynomial $f(x)$ can be expressed as $f(x) = Q(x)g(x) + R(x)$ where $Q(x)$ is the quotient and $R(x)$ is the remainder when $f(x)$ is divided by $g(x) = (x-\alpha)(x-\beta)$

(I) Show that $R(x) = \dfrac{(x-\beta)f(\alpha) + (\alpha-x)f(\beta)}{\alpha-\beta}$

(II) Given that when $f(x)$ is divided by $(x-3)$ the remainder is 2 and when divided by $(x+3)$ the remainder is -3, use the expression in (i) above to determine the remainder when $f(x)$ is divided by x^2-9.

Soln

(i) let $R(x) = ax+b$

$\Rightarrow f(x) = \varphi(x)[(x-\alpha)(x-\beta)] + ax+b$

Let $x=\alpha$

$\Rightarrow f(\alpha) = a\alpha+b$ - - - - - - - (i)

let $x=\beta$

$\Rightarrow f(\beta) = a\beta+b$ - - - - - - - (ii)

take (i) $-$ (ii)

$\Rightarrow a = \dfrac{f(\alpha)-f(\beta)}{\alpha-\beta}$ and $b = \dfrac{\alpha f(\beta)-\beta f(\alpha)}{\alpha-\beta}$

$\therefore R(x) = \dfrac{(x-\beta)f(\alpha) + (\alpha-x)f(\beta)}{\alpha-\beta}$

(ii) $x^2-9 = (x-3)(x+3)$

$\Rightarrow \alpha = 3$ and $\beta = -3$

But $f(\alpha)=2$ and $f(\beta) = -3$

Hence $R(x) = \dfrac{(x+3)(2) + (3-x)(-3)}{3--3}$

$= \dfrac{5x-3}{6}$

$= \dfrac{5}{6}x - \dfrac{1}{2}$

$\therefore a = 5/6$ and $b = -\frac{1}{2}$

The principle of the undetermined coefficients

This principle is applied in the determination of coefficients of polynomials by equating the coefficients of the corresponding terms.

Examples

(1) Given that $2x^2-9x+14 = a(x-1)(x-2)+b(x-1)+c$, find the values of a, b and c

Soln

$2x^2-9x+14 = a(x^2-3x+2)+bx-b+c$

$= ax^2 - (3a-b)x + 2a - b + c$

$\therefore a = 2,\ b = -3$ and $c = 5$

② Use the principle of the undetermined coefficients to find the square root of $x^4 + 4x^3 + 8x^2 + 8x + 4$.

Soln

Let $x^4 + 4x^3 + 8x^2 + 8x + 4 = (ax^2 + bx + c)^2$

On expanding $(ax^2 + bx + c)^2$ and comparing coefficients,

$a = 1,\ b = 2$ and $c = 2$

$\therefore$ The square root of $x^4 + 4x^3 + 8x^2 + 8x + 4$ is $x^2 + 2x + 2$

Exercise

① Divide $x^3 - 2x^2 + 5x + 8$ by $x - 2$

② Find the remainder when $x^5 + x - 9$ is divided by $x + 1$

③ If the expression $ax^4 + bx^3 - x^2 + 2x + 3$ has a remainder $3x + 5$ when divided by $x^2 - x - 2$, find the values of a and b

④ Find the values of p and q that make $x^4 + 6x^3 + 13x^2 + px + q$ a perfect square.

⑤ Factorise $3x^3 - 11x^2 - 19x - 5$ completely

⑥ Given that $(x+3)$ and $(x+7)$ are factors of $ax^2 + 12x + b$, find a and b

⑦ $f(x) = 2x^3 + ax^2 - 6x + 1$. If the remainder when $f(x)$ is divided by $x + 2$ is twice that when $f(x)$ is divided by $(x - 1)$, find the value of a.

PARTIAL FRACTIONS

A partial fraction is one of the constituents of a fraction that has been partialised.

TYPES OF FRACTIONS

(I) Fractions with a denominator having linear factors

These are fractions in which all the factors in the denominator are linear.

They are of the form $\frac{ax+b}{(a_1x+b_1)(a_2x+b)\cdots(a_nx+b_n)}$

To partialise such fractions we let

$$\frac{ax+b}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} \equiv \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

Where $A_1, A_2, \ldots, A_{n-1}$ and A_n are constants

Examples

① Express the following as partial fractions

(I) $\frac{11x+12}{(2x+3)(x+2)(x-3)}$ (II) $\frac{x-1}{3x^2-11x+10}$

Soln

(II) Let $\frac{x-1}{3x^2-11x+10} \equiv \frac{A}{x-2} + \frac{B}{3x-5}$

$\Rightarrow x-1 = A(3x-5) + B(x-2)$

when $x=2$, $1 = A+0$; $A=1$

when $x=0$, $-1 = -5-2B$; $B=-2$

$\therefore \frac{x-1}{3x^2-11x+10} = \frac{1}{x-2} - \frac{2}{3x-5}$

(I) $\frac{11x+12}{(2x+3)(x+2)(x-3)} = \frac{2}{2x+3} - \frac{2}{x+2} + \frac{1}{x-3}$

(II) Fractions with repeated factors in the denominator

These are fractions in which at least one of the factors in the denominator is raised to a power greater than one.

To partialise such fractions we let the component $\frac{1}{(ax+b)^n} \equiv \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$

Examples:

① Express the following as partial fractions.

(I) $\dfrac{4x+3}{(x-1)^2}$ (II) $\dfrac{1}{(x+2)(x-1)^2}$

Soln

(I) Let $\dfrac{4x+3}{(x-1)^2} \equiv \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$\Rightarrow 4x+3 = A(x-1) + B$

When $x=1$, $4+3 = 0+B$; $B = 7$

When $x=0$, $3 = -A+7$; $A = 4$

$\therefore \dfrac{4x+3}{(x-1)^2} = \dfrac{4}{(x-1)} + \dfrac{7}{(x-1)^2}$

Alternatively;

$$\frac{4x+3}{(x-1)^2} = \frac{4(x-1)+7}{(x-1)^2}$$

$$= \frac{4(x-1)}{(x-1)^2} + \frac{7}{(1-x)^2}$$

$$\therefore \frac{4x+3}{(x-1)^2} = \frac{4}{(x-1)} + \frac{7}{(x-1)^2}$$

(II) $\dfrac{1}{(x+2)(x-1)^2} = \dfrac{1}{9(x+2)} - \dfrac{1}{9(x-1)} + \dfrac{1}{3(x-1)}$

(III) Fractions with a quadratic factor in the denominator

These are fractions in which at least one of the factors in the denominator is a quadratic expression that can not be completely factorised.

To partialise such fractions we let the component $\dfrac{1}{ax^2+bx+c} \equiv \dfrac{Ax+B}{ax^2+bx+c}$

Examples

① Express the following as partial fractions

(I) $\dfrac{3x+1}{(x-1)(x^2+1)}$ (II) $\dfrac{4}{(x+1)(2x^2+x+3)}$

Soln

(i) Let $\dfrac{3x+1}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\therefore \dfrac{3x+1}{(x-1)(x^2+1)} = \dfrac{2}{x-1} + \dfrac{1-2x}{x^2+1}$

(ii) Let $\dfrac{4}{(x+1)(2x^2+x+3)} \equiv \dfrac{A}{x+1} + \dfrac{Bx+C}{2x^2+x+3}$

$\therefore \dfrac{4}{(x+1)(2x^2+x+3)} = \dfrac{1}{x+1} + \dfrac{1-2x}{2x^2+x+3}$

(iv) Improper fractions

These are fractions where the degree of the numerator is greater than or equal to the degree of the denominator.

To partialise such fractions we first divide the numerator by the denominator to obtain a quotient and a proper fraction which is then split up into its partial fractions.

Examples

① Express $\dfrac{x^3-x^2-4x+1}{x^2-4}$ as a partial fraction

Soln

By long division;

$$\frac{x^3-x^2-4x+1}{x^2-4} = x-1 - \frac{3}{(x-2)(x+2)}$$

Let $\dfrac{3}{(x-2)(x+2)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$$\therefore \frac{x^3-x^2-4x+1}{x^2-4} = x-1 - \frac{3}{4(x-2)} + \frac{3}{4(x+2)}$$

NB:

Partial fractions can be used in integration summing up series and to obtain the Binomial expansions of certain expressions.

Exercise

Express the following fractions as partial fractions

① (a) $\dfrac{3x+5}{(x+3)(x-1)}$ (b) $\dfrac{8x^2+13x+6}{(x+2)(2x+1)(3x+2)}$

(c) $\dfrac{5x+2}{(x+1)(x^2-4)}$ (d) $\dfrac{20x+84}{(x+5)(x^2-9)}$

(e) $\dfrac{1}{x^3-9x}$ (g) $\dfrac{6-14x-7x^2}{(1-4x)(1-x)(2+3x)}$

② (a) $\dfrac{6-x}{(x+1)(4+x)^2}$ (b) $\dfrac{3+2x}{(2-x)(3+x)^2}$

(c) $\dfrac{3x+7}{(x+2)(x+3)^2}$ (d) $\dfrac{2x^2-5x+7}{(x+1)(x-1)^2}$

(e) $\dfrac{x+1}{(x+3)^3}$ (f) $\dfrac{32x^2+17x+18}{(2-3x)(1+2x)^2}$

③ (a) $\dfrac{11x+3}{(1+x^2)(1-5x)}$ (b) $\dfrac{5x^2-2x-1}{(x+1)(x^2+1)}$

(c) $\dfrac{2x^3+11}{(x^2+4)(x-3)}$ (d) $\dfrac{18-x-x^2}{(1+x+x^2)(2-3x)}$

(e) $\dfrac{2x^2-x+3}{(x+1)(x^2+2)}$ (f) $\dfrac{x^2+2x+18}{x(x^2+3)^2}$

④ (a) $\dfrac{3x^2-2x-7}{(x-2)(x+1)}$ (b) $\dfrac{x^3+2x-2x+2}{(x-1)(x+3)}$

(c) $\dfrac{x^4+3x-1}{(x+2)(x-1)^2}$ (d) $\dfrac{x^4-2x^3-x^2-4x+4}{(x-3)(x^2+1)}$

(e) $\dfrac{2x^4-17x-1}{(x-2)(x^2+5)}$ (f) $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}$