VECTORS

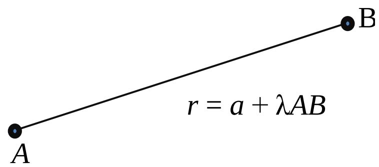
Straight line in space

A straight line is uniquely determined in space if either; we know one point on the straight line and its direction or two points on the straight line.

Vector equation of a line

The vector equation of a line is given by

$r = \boldsymbol{a} + \lambda AB$



$r = \boldsymbol{a} + \lambda AB$

$r = a + \lambda d$

Where; a = any point on the line

d = directional vector of the line.

> The Cartesian equation is given by;
> $$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} = \lambda$$
> Where a, b and c are direction vectors

Example 1

Find the vector and Cartesian equation of a line passing through $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and is parallel to $3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

Solution;

$r = a + \lambda d$

$$r = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Cartesian equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$x = 3 + 3\lambda$

$y = -1 - \lambda$

$z = 2 + 2\lambda$

$$\frac{x-3}{3} = \lambda, \frac{y+1}{-1} = \lambda, \frac{z-2}{2} = \lambda$$

$$\frac{x-3}{3} = \frac{y+1}{-1} = \frac{z-2}{2} = \lambda$$

Example II

Find the vector and the Cartesian equation of a line passing through A(3, 4, -7) and B(1, -1, 6)

Solution

$\boldsymbol{r} = a + \lambda d$

$d = AB = OB - OA$

$$\begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix} \text{(vector equation of line)}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -5 \\ 13 \end{pmatrix}$$

$x = 3 - 2\lambda$

$y = 4 - 5\lambda$

$z = -7 + 13\lambda$

$$\frac{x-3}{-2} = \lambda$$

$$\frac{y-4}{-5} = \lambda$$

$$\frac{z+7}{13} = \lambda$$

Cartesian equation

$$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13} = \lambda$$

Example III

Find the vector and Cartesian equation of a line passing through (2, -1, 1) and is parallel to the line whose equation

$$\frac{x-3}{2} = \frac{y+1}{7} = \frac{z-2}{-3} = \lambda$$

Solution

Since the lines are parallel, it implies that they have the same parallel vectors.

$\boldsymbol{r} = a + \lambda d$

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

Cartesian equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$x - 2 = 2\lambda \Rightarrow \frac{x-2}{2} = \lambda$$

$$y + 1 = 7\lambda \Rightarrow \frac{y+1}{7} = \lambda$$

$z - 1 = -3\lambda \Rightarrow \frac{z-1}{-3} = \lambda$

$$\frac{x-2}{2} = \frac{y+1}{7} = \frac{z-1}{-3} = \lambda$$

Example III

Find the vector and Cartesian equations of the a line passing through the following points

(a) 5, -4, 6) and (3, 7, 2)

(b) (3, 4, -7) and (5, 1, 6)

Solution

r = a + λAB

r = a + λd

$$\boldsymbol{d} = \begin{pmatrix}3\\7\\2\end{pmatrix} - \begin{pmatrix}5\\-4\\6\end{pmatrix} = \begin{pmatrix}-2\\11\\-4\end{pmatrix}$$

(a) $\boldsymbol{r} = \begin{pmatrix}5\\-4\\6\end{pmatrix} - \lambda\begin{pmatrix}-2\\11\\-4\end{pmatrix}$

$$\frac{x-5}{-2} = \frac{y+4}{11} = \frac{z-6}{-4} = \lambda$$

(b) A(3, 4, -7) and B(5, 1, 6)

r = a + μd

$$\boldsymbol{d} = \begin{pmatrix}5\\1\\6\end{pmatrix} - \begin{pmatrix}3\\4\\-7\end{pmatrix}$$

$$\boldsymbol{d} = \begin{pmatrix}2\\-3\\13\end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix}3\\4\\-7\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\13\end{pmatrix}$$

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z+7}{13} = \lambda$$

Example IV

Find the coordinates of the point where the line joining the points (2, 3, 1) and (3, -4 -5) meets the *x*-*y* plane

$$\boldsymbol{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-7\\-6\end{pmatrix}$$

$x = 2 + \lambda$

$y = 3 - 7\lambda$

$z = 1 - 6\lambda$

For the line to meet the *x*-*y* plane, *z* = 0

$0 = 1 - 6\lambda$

$\lambda = \frac{1}{6}$

$x = 2 + \frac{1}{6}$

$x = \frac{13}{6}$

$y = 3 - \frac{7}{6}$

$y = \frac{11}{6}$

The coordinates are $\left(\frac{13}{6}, \frac{11}{6}, 0\right)$

Example V

Show that 4**i** – **j** – 12**k** lies on the line

$$\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k} + \lambda(\boldsymbol{i} - \boldsymbol{2}j + 4\boldsymbol{k}$$

Solution

$$\boldsymbol{r} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\4\end{pmatrix}$$

$(4, -1, 12)$

$$\begin{pmatrix}4\\-1\\12\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\4\end{pmatrix}$$

$4 = 2 + \lambda \Rightarrow \lambda = 2$

$-1 = 3 - 2\lambda \Rightarrow \lambda = 2$ and

$12 = 4 + 4\lambda \Rightarrow \lambda = 2$

∴ The point lies on the line since the values of μ are the same.

Example V

The points A, B, C have position vectors $\begin{pmatrix}-4\\5\\-1\end{pmatrix}, \begin{pmatrix}5\\2\\3\end{pmatrix}, \begin{pmatrix}8\\1\\7\end{pmatrix}$. Find which of the three points lie in the line $\boldsymbol{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$

Solution

$$\boldsymbol{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

For *A*, $\boldsymbol{r} = \begin{pmatrix}-4\\5\\-1\end{pmatrix}$

$$\begin{pmatrix}-4\\5\\-1\end{pmatrix} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

$-4 = -1 + 3\lambda \Rightarrow \lambda = -1$

$5 = 4 - \lambda \Rightarrow \lambda = -1$

$-1 = 1 + 2\lambda \Rightarrow \lambda = -1$

$\Rightarrow \begin{pmatrix}-4\\5\\-1\end{pmatrix}$ lies on the line.

For B, $\begin{pmatrix}5\\2\\3\end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

$$\begin{pmatrix}5\\2\\3\end{pmatrix} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

5 = -1 + 3λ ⇒ λ = 2

2 = 4 – λ ⇒ λ = 2

3 = 1 + 2λ ⇒ λ = 1

Since the values of λ are not the same, point B does not lie on the line.

For C, $\begin{pmatrix}8\\1\\7\end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

$$\begin{pmatrix}8\\1\\7\end{pmatrix} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

8 = -1 + 3λ ⇒ λ = 3

1 = 4 – λ ⇒ λ = 3

7 = 1 + 2λ ⇒ λ = 3

⇒ Since the vales of λ are the same, point C lies on the line.

Angle between two lines

The angle between two lines is the angle between their directional vectors

Consider two lines L_1 and L_2 with vector equations
$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r} = \boldsymbol{b} + \mu \boldsymbol{d}_2$ respectively
The angle between the two lines is given by the formula $\dfrac{d_1 \cdot d_2}{|d_1||d_2|}$

Examples

1. Find the angle between the lines;

$\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k} + \lambda(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$

$\boldsymbol{r} = 5\boldsymbol{i} - 2\boldsymbol{j} + \mu(3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\boldsymbol{\theta}$

$$\cos\boldsymbol{\theta} = \frac{a.b}{|a||b|}$$

$$\cos\theta = \frac{d_1.d_2}{|d_1||d_2|}$$

$$d_1 = \begin{pmatrix}1\\2\\2\end{pmatrix} \qquad d_2 = \begin{pmatrix}3\\2\\6\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\2\end{pmatrix}.\begin{pmatrix}3\\2\\6\end{pmatrix}}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}}$$

$$\cos\theta = \frac{3+4+12}{\sqrt{9}\sqrt{49}}$$

$$\cos\theta = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

$$\theta = 25.2°$$

Example II

Find the angles between the lines

$$\frac{x+4}{3} = \frac{y+1}{5} = \frac{z+3}{4} \;\&\; \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution

$$\cos\theta = \frac{d_1.d_2}{|d_1||d_2|}$$

$$d_1 = \begin{pmatrix}3\\5\\4\end{pmatrix}, \quad d_2 = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}3\\5\\4\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix}}{\sqrt{3^2+5^2+4^2}\sqrt{1^2+1^2+2^2}}$$

$$\cos\theta = \frac{3+5+8}{(\sqrt{50})\sqrt{6}}$$

$$\cos\theta = \frac{16}{\sqrt{300}}$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{300}}\right)$$

$$\theta = 22.5°$$

Example III

Find the acute angle between the lines:

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \text{ and } \frac{1-x}{2} = \frac{y-3}{1} = \frac{z-7}{2}$$

Solution

$$\Rightarrow \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-2}{-1} \text{ and } \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\Rightarrow \frac{x-1}{-2} = \frac{y-3}{1} = \frac{z-7}{2}$$

$$\boldsymbol{d_1} = \begin{pmatrix}2\\1\\-1\end{pmatrix} \text{ and } \boldsymbol{d_2} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{d_1 \cdot d_2}{|d_1| \cdot |d_2|}$$

$$\cos\theta = \frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix}}{\sqrt{2^2+1^2+(-1)^2}\sqrt{(-2)^2+1^2+(2)^2}}$$

$$\cos\theta = \frac{-4+1-2}{\sqrt{6}\,.\sqrt{9}}$$

$\Rightarrow$ The acute angle between the two lines is 47.1°

Example IV

Find the angle between the lines:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{1} = \lambda \text{ and } \frac{x-5}{1} = \frac{y-1}{1} = \frac{z}{2} = \mu$$

Solution

$$\boldsymbol{d_1} = \begin{pmatrix}3\\2\\1\end{pmatrix} \text{ and } \boldsymbol{d_2} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\cos\theta = \frac{d_1 \cdot d_2}{|d_1| \cdot |d_2|}$$

$$\cos\theta = \frac{\begin{pmatrix}3\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}}{\sqrt{3^2+2^2+(1)^2}\sqrt{(1)^2+1^2+(2)^2}}$$

$$\cos\theta = \frac{3+2+2}{\sqrt{14}\,.\sqrt{6}}$$

$\theta = 40.2°$

$\Rightarrow$ The acute angle between the two lines is 40.2°

Note: If two lines are perpendicular, then $(\boldsymbol{d_1}.\boldsymbol{d_2}) = \boldsymbol{0}$

Point of Intersection of two Lines

Example

Find the point of intersection of the lines

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} \;\&\; \frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$$

Solution

$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} = \lambda$ ………………. (i)

$\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1} = \mu$ ……………. (ii)

From equation (i)

$x = \lambda$ ………………………………. (iii)

$\frac{y+2}{2} = \lambda$

$y + 2 = 2\lambda$

$y = 2\lambda - 2$ ………………………. (iv)

$\frac{z-5}{-1} = \lambda$

$z - 5 = -\lambda$

$z = -\lambda + 5$ ………………………. (v)

From equation (ii)

$x = -\mu + 1$ ………………………. (vi)

$y + 3 = -3\mu$

$y = -3\mu - 3$ … … … … … … … … … … (vii)

$z = \mu + 4$ ………………………….. (viii)

$\lambda = -\mu + 1$ … … … … … … … … … … (*)

$2\lambda - 2 = -3\mu - 3$

$2\lambda + 3\mu = -1$ … … … … … … … … . . (**)

Substituting Eqn (*) in Eqn (**)

$2(1-\mu) + 3\mu = -1$

$2 - 2\mu + 3\mu = -1$

$2 + \mu = -1$

$\mu = -3$

$\lambda = -\mu + 1$

$\lambda = 3 + 1$

$\lambda = 4$

Equating Eqn (v) and Eqn (viii)

$-\lambda + 5 = \mu + 4$

$-4 + 5 = -3 + 4$

$1 = 1$

The two lines intersect

$x = 4$

$y = 2\lambda - 2$

$y = 8 - 2$

$y = 6$

$z = -4 + 5$

$z = -4 + 5 = 1$

The point of intersection of the lines is (4, 6, 1)

Example II

Find the point of intersection of the line

$\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k} + \mu(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

Solution

From $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$$\boldsymbol{r} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$\begin{pmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\end{pmatrix} = \begin{pmatrix}1+2\lambda\\-2+\lambda\\3-\lambda\end{pmatrix}$…………………… (1)

$\boldsymbol{r} = -\boldsymbol{i} + 3\boldsymbol{j} + 7\boldsymbol{k} + \mu(-2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$

$$\boldsymbol{r} = \begin{pmatrix}-1\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

$$(\boldsymbol{r}) = \begin{pmatrix}-1-2\mu\\3+\mu\\1+2\mu\end{pmatrix}$$

$$\begin{pmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\end{pmatrix} = \begin{pmatrix}-1-2\mu\\3+\mu\\7+2\mu\end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equating the corresponding *x* components:

$1 + 2\lambda = -1 - 2\mu$

$2\lambda + 2\mu = -2$

$\lambda + \mu = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Equating the corresponding *y* components:

$-2 + \lambda = 3 + \mu$

$\lambda - \mu = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Equating the corresponding *z* component;

$3 - \lambda = 7 + 2\mu$

$2\mu + \lambda = -4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$

Eqn (3) –eqn (4)

$2\mu = -6$

$\mu = -3$

From Eqn (4)

$\lambda - (-3) = 5$

$\lambda = 2$

Substituting $\lambda = 2$ and $\mu = -3$ in Eqn (5);

$\Rightarrow$ The two lines intersect at (5, 0, 1)

Example III

Find the point of intersection of the lines

$$x - 2 = \frac{y+3}{4} = \frac{z-5}{2} \;\&\; \frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2}$$

Solution

$$x - 2 = \frac{y+3}{4} = \frac{z-5}{2} = \lambda \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

$$\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2} = \mu \ldots\ldots\ldots\ldots\ldots\ldots (**)$$

From equation (*)

$x - 2 = \lambda$

$x = 2 + \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$y + 3 = 4\lambda$

$y = 4\lambda - 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

$z - 5 = 2\lambda$

$z = 2\lambda + 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From equation (**)

$x - 1 = -\mu$

$x = 1 - \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

$y - 8 = \mu$

$y = \mu + 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$

$z - 3 = 2\mu$

$z = 2\mu + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$

Equating the corresponding components

$2 + \lambda = 1 - \mu$

$\mu + \lambda = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$

$\mu + 8 = 4\lambda - 3$

$\mu - 4\lambda = -11 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$

Eqn(8) - (7)

$-5\lambda = -10$

$\lambda = 2$

Substitute λ = 2 in Eqn (8)

$\mu - 4 \times 2 = -11$

$\mu = -3$

∴ The point of intersection is (4, 5, 9)

PLANES

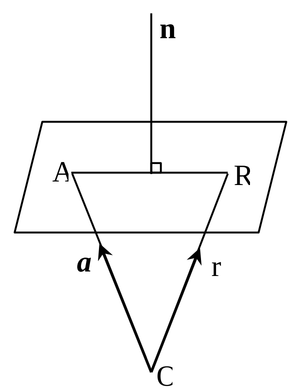
A plane is a surface which contains at least three non-collinear points. If two points are taken then the lines joining the two lines lies completely on the surface of the plane.

A plane is completely known if we know one point that lie on the plane and then the normal to the plane.

Equation of a Plane

Suppose a plane P passes through a point A with a position vector ***a*** and is perpendicular to vector **n**. Let **r** be any point (*x*, *y*, *z*) in the plane.

If two lines are perpendicular, dot product of their direction vector = 0



$AR.n = 0$

$(AO + OR).n = 0$

$(-\boldsymbol{a} + \boldsymbol{r}).n = 0$

$(-n.\boldsymbol{a} + n.\boldsymbol{r}) = 0$

$n.\boldsymbol{a} = n.\boldsymbol{r}$

Equation of a plane is given by $\mathbf{n}.\boldsymbol{r} = \mathbf{n}.\boldsymbol{a}$

Where **n** = normal and **a** = the point that lies on the plane.

Example I

Find the equation of a plane passing through (1, 2, 3), and is perpendicular to vector $\boldsymbol{4i} + 5\boldsymbol{j} + 6\boldsymbol{k}$

Solution

$$\mathbf{n}.\mathbf{r} = \mathbf{n}.\boldsymbol{a}$$
$$\begin{pmatrix}4\\5\\6\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix}$$
$$4x + 5y + 6z = 4 + 10 + 18$$
$$4x + 5y + 6z = 32$$

Example II

Find the equation of a plane which contains A with position vector $\boldsymbol{3i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ and is perpendicular to $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$**.**

Solution

$$n.\boldsymbol{r} = n.\boldsymbol{a}$$
$$\begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}3\\4\\2\end{pmatrix}$$
$$x + 2y - 2z = 3 + 8 - 4$$
$$x + 2y - 2y = 7$$

Example III

Find the equation of a plane passing through a point A with a position vector $-2\boldsymbol{i} + 4\boldsymbol{k}$ and is perpendicular to the vector $\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$**.**

Solution

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\boldsymbol{a}$$
$$\begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}-2\\0\\4\end{pmatrix}$$
$$x + 3y - 2z = -2 + 0 - 8$$
$$x + 3y - 2z = -10$$
$$x + 3y - 2z + 10 = 0$$

Angle between two planes

The angle between two planes is the angle between their normals

$$\cos\theta = \frac{n_1.n_2}{|n_1||n_2|}$$

Example I

Find the angle between the planes $2x + 3y + 5z = 7$, $3x + 4y - z = 8$

Solution

$$n_1 = \begin{pmatrix}2\\3\\5\end{pmatrix}, n_2 = \begin{pmatrix}3\\4\\-1\end{pmatrix}$$
$$\cos\theta = \frac{n_1.n_2}{|n_1||n_2|}$$
$$\cos\theta = \frac{\begin{pmatrix}2\\3\\5\end{pmatrix}\begin{pmatrix}3\\4\\-1\end{pmatrix}}{\sqrt{2^2+3^2+5^2}.\sqrt{3^2+4^2+1^2}}$$
$$\cos\theta = \frac{6+12-5}{\sqrt{38}.\sqrt{26}} = \frac{13}{\sqrt{38}.\sqrt{26}}$$
$$\theta = \cos^{-1}\frac{13}{\sqrt{38}.\sqrt{26}}$$
$$\theta = 65.6°$$

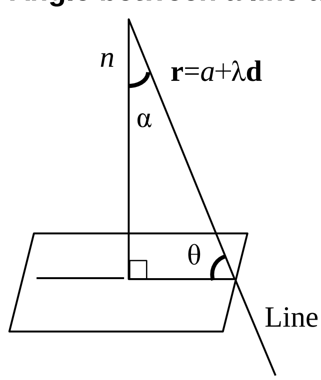
Example II

Find the angle between the planes $3x – 3y – z = 0$ and $x + 4y - 2z = 4$

Solution

$$n_1 = \begin{pmatrix}3\\-3\\-1\end{pmatrix}, n_2 = \begin{pmatrix}1\\4\\-2\end{pmatrix}$$
$$\cos\theta = \frac{n_1 n_2}{|n_1||n_2|}$$
$$\cos\theta = \frac{\begin{pmatrix}3\\-3\\-1\end{pmatrix}\begin{pmatrix}1\\4\\-2\end{pmatrix}}{\sqrt{3^2+(-3)^2+(-1)^2}.\sqrt{1^2+4^2+(-2)^2}}$$
$$\cos\theta = \frac{3-12+2}{\sqrt{19}.\sqrt{21}} = \frac{-7}{\sqrt{21}.\sqrt{19}}$$
$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{21}.\sqrt{19}}\right)$$
$$\theta = 69.5°$$

Angle between a line and a plane



$$\mathrm{n.d} = |n||d|\cos\alpha$$
$$\theta + 90° + \alpha = 180°$$
$$\theta + \alpha = 90°$$
$$\alpha = 90° - \theta$$

$\text{n.d} = |n||d|\cos(90° - \theta)$

$\text{n.d} = |n||d|\sin\theta$

$\sin\theta = \dfrac{n.d}{|n||d|}$

$$\boldsymbol{\sin\theta = \frac{n.d}{|n||d|}}$$

Example

Find the angle between the lines $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ and the plane $2x - y + z = 4$

Solution

$\sin\theta = \dfrac{n.d}{|n||d|}$

$\sin\theta \dfrac{\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix}}{\sqrt{1^2+(-1)^2+1^2}.\sqrt{2^2+(-1)^2+1^2}}$

$\sin\theta = \dfrac{2+1+1}{\sqrt{3}.\sqrt{6}}$

$\sin\theta = \left(\dfrac{4}{\sqrt{18}}\right)$

$\theta = \sin^{-1}\left(\dfrac{4}{\sqrt{18}}\right)$

$\theta = 70.5°$

Find the acute angle between the line $\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-2}$ and $7x - y + 5z = -5$

Solution

$\sin\theta = \dfrac{n.d}{|n||d|}$

$\sin\theta \dfrac{\begin{pmatrix}5\\-1\\1\end{pmatrix}\begin{pmatrix}7\\-1\\5\end{pmatrix}}{\sqrt{5^2+(-1)^2+1^2}.\sqrt{7^2+(-1)^2+5^2}}$

$\sin\theta = \dfrac{35+1+5}{\sqrt{27}.\sqrt{75}}$

$\sin\theta = \left(\dfrac{41}{\sqrt{2025}}\right)$

$\theta = \sin^{-1}\left(\dfrac{41}{\sqrt{2025}}\right)$

$\theta = 65.7°$

Solution

Find the angle between the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and $x + y + z = 12$

Solution

$\sin\theta = \dfrac{n.d}{|n||d|}$

$\sin\theta \dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\5\\-1\end{pmatrix}}{\sqrt{1^2+1^2+1^2}.\sqrt{2^2+5^2+1^2}}$

$\sin\theta = \dfrac{2+5-1}{\sqrt{3}.\sqrt{30}}$

$\sin\theta = \left(\dfrac{6}{\sqrt{90}}\right)$

$\theta = \sin^{-1}\left(\dfrac{6}{\sqrt{90}}\right)$

$\theta = 39.2°$

Point of intersection of a line and a plane

Example I

Find the point of intersection of the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and $x + y + z = 19$

Solution

$\dfrac{x+1}{5} = \dfrac{y-3}{-1} = \dfrac{z+1}{1} = \lambda$ ………………. (*)

From (*)

$x + 1 = 2\lambda$

$x = 2\lambda - 1$……………….. (1)

$y - 3 = 5\lambda$

$y = 3 + 5\lambda$……………….…. (2)

$z + 1 = -\lambda$

$z = -1 - \lambda$……………….... (3)

$x + y + z = 12$

$(2\lambda - 1) + (5 + 3\lambda) + (-\lambda - 1) = 12$

$4\lambda = 16$

$\lambda = 4$

From equation (1)

$x = 2(4) - 1 = 7$

From equation (2)

$y = 5(4) + 3 = 23$

From equation (3)

$z = -1 - 4 = -5$

∴ The point of intersection $(7, 23, -5)$

Example II

Find the point of intersection of the line $\dfrac{x}{5} = \dfrac{y+2}{2} = \dfrac{z-1}{4}$ and the plane $3x + 4y + 2z = 25$

Solution

$\dfrac{x}{5} = \dfrac{y+2}{2} = \dfrac{z-1}{4} = \lambda$ ……………….. (*)

$x = 5\lambda$………………………….. (1)

$y+2=2\lambda$
$y=2\lambda-2$……………………... (2)
$z-1=4\lambda$
$z=4\lambda+1$…………………….. (3)
$3x+4y+2z=25$
$3(5\lambda)+4(2\lambda-2)+2(4\,\lambda+1)=25$
$15\lambda+8\lambda-8+8\lambda+2=25$
$31\lambda=25+6$
$31\lambda=31$
$\lambda=1$
$x=5,\quad y=2-2=0,\quad z=5$
∴ The point of intersection = (5, 0, 5)

Example
Find the point of intersection of the line; $\frac{x+2}{-1}=\frac{y-2}{2}=z-4$ and the plane $2x-y+3z=10$
Solution
$$\frac{x+2}{-1}=\frac{y-2}{2}=z-4=\lambda$$
$x=-\lambda-2$…………….. (1)
$y=2\lambda+2$……………... (2)
$z=\lambda+4$………………. (3)
$2x-y+3z=10$
$2(-\lambda-2)-(2\lambda+2)+3(\lambda+4)=10$
$-2\lambda-4-2\lambda-2+3\lambda+12=10$
$-4\lambda+3\lambda+6=10$
$-\lambda=4$
$\lambda=-4$
$x=-4-2=-6,\quad y=-8+2=-6,$
$z=-4+4=0$
The point of intersection (-6, -6, 0)

Perpendicular distance of a point from a plane

The perpendicular distance of a point (x_1,y_1,z_1) from the plane $ax+by+cz+d=0$ is given by the formula;
$$D=\left|\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right|$$
Example
Find the distance of a point $(-2,0,6)$ from the plane $2x-y+3z=21$
Solution
$$D=\left|\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right|$$
$x_1,y_1,z_1=(-2,0,6)$
Comparing $ax+by+cz+d=0$ with
$2x-y+3z-21=0;$
$a=2,\quad b=-1,\quad c=3,\quad d=-21$
$$D=\left|\frac{-4+0+18-21}{\sqrt{2^2+(-1)^2+3^2}}\right|$$
$D=\frac{-7}{\sqrt{4+1+9}}=\frac{-7}{\sqrt{14}}$ Units

Line of intersection of two planes

Two planes intersect in a line

Examples I
Find the line of intersection of the planes $2x+3y+4z=1$ and $x+y+3z=0$
Solution
$2x+3y+4z=1$
$x+y+3z=0$
Let $z=\lambda$
$2x+3y=1-4\lambda$…………………….. (1)
$x+y=-3\lambda$ ……………………………..(2)
Eqn (2) ×2
$2x+2y=-6\lambda$ ……………………… (3)
Eqn (1) – Eqn (3);
$y=1+2\lambda$
$$\frac{y-1}{2}=\lambda$$
From Eqn (2);
But $y=1+2\lambda$
$x+y=-3\lambda$
$x+1+2\lambda=-3\lambda$
$x+1=-3\lambda-2\lambda$
$x+1=-5\lambda$
$$\frac{x+1}{-5}=\lambda$$
$$\frac{x+1}{-5}=\frac{y-1}{2}=z=\lambda$$

Example II
Find the line of intersection of planes $2x+3y-z=4$ and $x-y+2z=5$.
Solution
$2x+3y-z=4$
$x-y+2z=5$
Let $z=\lambda$
$2x+3y-\lambda=4$
$x-y+2\lambda=5$
$2x+3y=4+\lambda$ ……………… (i)
$x-y=5-2\lambda$……………….. (ii)

Multiply Eqn (ii) by 3;

$3x - 3y = 15 - 6\lambda$ …………… (iii)

Eqn (iii) + Eqn (i);

$5x = 19 - 5\lambda$

$5\lambda = -x + 19$

$\lambda = -x + \frac{19}{5}$

$\lambda = \frac{\left(x - \frac{19}{5}\right)}{-1}$

Multiply Eqn (ii) by 2;

$2x - 2y = 10 - 4\lambda$ …………… (iv)

Eqn (iv) – Eqn (i);

$-5y = 6 - 5\lambda$

$5\lambda = -6 + 5y$

$\lambda = \frac{-6}{5} + y$

$\lambda = \frac{y - \frac{6}{5}}{1}$

$\frac{x - \frac{19}{5}}{-1} = \frac{y - \frac{6}{5}}{1} = z = \lambda$

Example

Find the Cartesian equation of a line of intersection of the lines.

$2x - 3y - z = 1$

$3x + 4y + 2z = 3$

Let $x = \lambda$

$-3y - z = 1 - 2\lambda$ …………… (i)

$4y + 2z = 3 - 3\lambda$ …………… (ii)

Eqn (i) × 2

$-6y - 2z = 2 - 4\lambda$ ………….. (iii)

Eqn (iii) + Eqn (ii)

$-2y = 5 - 7\lambda$

$-2y - 5 = -7\lambda$

$\frac{-2y - 5}{-7} = \lambda$

$\frac{-2\left(y + \frac{5}{2}\right)}{-7} = \lambda$

Eqn (i) × 4

$\Rightarrow -12y - 4z = 4 - 8\lambda$ ……….. (iv)

Eqn (ii) × 3

$12y + 6z = 9 - 9\lambda$ ………………. (v)

Eqn (iv) + Eqn (v)

$2z = 13 - 17\lambda$

$\frac{2z - 13}{-17} = \lambda$

$\frac{2\left(z - \frac{13}{2}\right)}{-17} = \lambda$

$x = \frac{\left(y + \frac{1}{2}\right)}{\frac{7}{2}} = -\frac{\left(z - \frac{13}{2}\right)}{\frac{17}{2}} = \lambda$

$x = \frac{\left(y + \frac{1}{2}\right)}{\frac{7}{2}} = \frac{\left(z - \frac{13}{2}\right)}{-\frac{17}{2}} = \lambda$

Equation of a Plane

Given three points on the plane, we can find the equation of a plane;

Example I

Find the Cartesian equation of a plane passing through A (0, 3, -4) B (2, -1, 2) and C (7, 4, -1)

Solution

Let the normal = $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$AB = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$

$AC = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = 0$

$2p - 4q + 6r = 0$

$p - 2q + 3r = 0$ …………………… (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} = 0$

$7p + q + 3r = 0$ …………………… (ii)

From (i)

$p = 2q - 3r$ ………………………… (iii)

$\Rightarrow 7(2q - 3r) + q + 3r = 0$

$14q - 21r + q + 3r = 0$

$15q - 18r = 0$

$5q - 6r = 0$

$5q = 6r$

$q = \frac{6}{5}r$……………………………… (iv)

$\Rightarrow p = 2\left(\frac{6r}{5}\right) - 3r$

$p = \frac{12}{5}r - 3r$

$p = -\frac{3}{5}r$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -3r/5 \\ 6r/5 \\ r \end{pmatrix} = \frac{r}{5}\begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$

$\therefore n = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$

$n.\boldsymbol{r} = n.\boldsymbol{a}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}.\begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$

$-3x + 6y + 5z = 0 + 18 - 20$

$-3x + 6y + 5z = -2$

$3x - 6y - 5z - 2 = 0$

Example II

Find the equation of a plane passing through points P(4, 2, 3), Q(5, 1, 4) and R(-2, 1, 1).

Solution

Let the normal to the plane be $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$PQ = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$PR = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix}.\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$

$p - q - r = 0$ …………………… (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix}.\begin{pmatrix} -6 \\ -1 \\ -2 \end{pmatrix} = 0$

$-6p - q - 2r = 0$

$6p + q + 2r = 0$ ………………… (ii)

From Eqn (i);

$p = q - r$

$6(q - r) + q + 2r = 0$

$6q - 6r + q + 2r = 0$

$7q - 4r = 0$

$7q = 4r$

$q = \frac{4r}{7}$

$p = \frac{4r}{7} - r$

$\mathrm{p} = \frac{-3\mathrm{r}}{7}$

$\begin{pmatrix} \mathrm{p} \\ \mathrm{q} \\ \mathrm{r} \end{pmatrix} = \begin{pmatrix} \frac{-3\mathrm{r}}{7} \\ \frac{4\mathrm{r}}{7} \\ \mathrm{r} \end{pmatrix} = \frac{\mathrm{r}}{7}\begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}$

$n = \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}$

$\mathrm{n.r} = \mathrm{n.a}$

$\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}\begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix}.\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

$-3\mathrm{x} + 4\mathrm{y} + 7\mathrm{z} = -12 + 8 + 21$

$-3\mathrm{x} + 4\mathrm{y} + 7\mathrm{z} = 17$

$3\mathrm{x} - 4\mathrm{y} - 7\mathrm{z} + 17 = 0$

Example III

Find the equation of the planes passing through the following points:

(i) A (0, 2, -4) B (2, 0, 2) C (-8, 4, 0)

Solution

Let the normal $n = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$AB = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$

$AC = \begin{pmatrix} -8 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ 4 \end{pmatrix}$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix}.\begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} = 0$

$2p - 2q + 6r = 0$

$p - q + 3r = 0$ …………………… (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix}.\begin{pmatrix} -8 \\ 2 \\ 4 \end{pmatrix} = 0$

$-8p + 2q + 4r = 0$

$-4p + q + 2r = 0$ …………………… (ii)

$p - q + 3r = 0$

$p = q - 3r$

$-8(q - 3r) + 2q + 4r = 0$

$-8q + 24r + 2q + 4r = 0$

$-6q + 28r = 0$

$6q = 28r$

$q = \frac{14r}{3}$

$p = \frac{14r}{3} - 3r = \frac{5r}{3}$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 5r/3 \\ 14r/3 \\ r \end{pmatrix} = \frac{r}{3}\begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix}$$

$$n = \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix}$$

$$n.\boldsymbol{r} = n.\boldsymbol{a}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix}$$

$$5x + 14y + 3z = 0 + 28 - 12$$

$$5x + 14y + 3z - 16 = 0$$

(ii) A (-1, 0, 1), B(3, 3, -2), C(-1, 1, 1)

Let the normal $= \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$\text{AB} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$$

$$AC = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} = 0$$

$4p + 3q - 3r = 0$ ………………… (i)

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$q = 0$$

Substitute $q = 0$ in Eqn (i);

$$4p = 3r$$

$$p = \frac{3r}{4}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 3r/4 \\ 0 \\ r \end{pmatrix} = \frac{r}{4}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$n = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$n.\boldsymbol{r} = n.\boldsymbol{a}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$3x + 4z = -3 + 4$$

$$(3x + 4z = 1)$$

$$3x + 4z - 1 = 0$$

Example IV

Find the Cartesian equation of a plane containing the point (1, 3, 1) and it's parallel to vectors (1, -1, -3) and (2, 1, -3)

Solution

$AB = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $AC = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

Let the normal $= \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0$$

$p - q + 3r = 0$ ……………….. (i)

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = 0$$

$2p + q - 3r = 0$ ……………… (ii)

$$p = q - 3r$$

$$2(q - 3r) + q - 3r = 0$$

$$2q - 6r + q - 3r = 0$$

$$3q - 9r = 0$$

$$q = 3r$$

$$p = 3r - 3r$$

$$p = 0$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 3r \\ r \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = r\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{a}$$

$$n = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$3y + z = 10$$

Example V

Find the Cartesian equation of the plane passing through the points A(1, 0, -2), B (3, -1, 1) parallel to the line

$\boldsymbol{r} = 3\boldsymbol{i} + (2\alpha - 1)\boldsymbol{j} + (5 - \alpha)\boldsymbol{k}$

Solution:

$$\boldsymbol{r} = 3\boldsymbol{i} + 2\alpha\boldsymbol{j} - \boldsymbol{j} + 5\boldsymbol{k} - \alpha\boldsymbol{k}$$

$$\boldsymbol{r} = 3\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k} - \alpha(\boldsymbol{0j} + \boldsymbol{2j} - \boldsymbol{k})$$

$$AB = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$AB = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$AC = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$

$2p – q + 3r = 0$ …………………….. (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 0$

$2q - r = 0$………………………… (ii)

From Eqn (ii);

$\Rightarrow r = 2q$

$2p - q + 3(2q) = 0$

$2p - q + 6q = 0$

$2p + 5q = 0$

$p = \frac{-5}{2} q$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{-5q}{2} \\ q \\ 2q \end{pmatrix} = \frac{q}{2} \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$

$n = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$

$n.\boldsymbol{r} = n.\boldsymbol{a}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$-5x + 2y + 4z = -5 - 8$

$(-5x + 2y + 4z = -13)$

$5x - 2y - 4z - 13 = 0$

Example VI

Find the equation of the plane containing line

$r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ and is parallel to the line

$r = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + S \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$AB = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad AC = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad n = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 0$;

$–2p + q – r = 0$

$\Rightarrow 2p – q + r = 0$ ……………….. (i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0$

$-p + q + 2r = 0$……………….. (ii)

From Eqn (i);

$r = –2p + q$

$\Rightarrow p – q – 2(q – 2p) = 0$

$p – q – 2q + 4p = 0$

$5p – 3q = 0$

$p = \frac{3q}{5}$

$r = -2\left(\frac{3q}{5}\right) + q$

$r = \frac{-q}{5}$

$n = \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{3q}{5} \\ q \\ \frac{-q}{5} \end{pmatrix} = \frac{q}{5} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

$n.\boldsymbol{r} = n.\boldsymbol{a}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$3x + 5y – z = 3 – 5 + 0$

$3x + 5y – z = -2$

Example VII

Find the Cartesian equation of the plane formed by the lines **r** = -2**i** + 5**j** – 11**k** + λ(3**i** + **j** + 3**k**) and **r** = 8**i** + 9**j** + λ(4**i** + 2**j** + 5**k**)

Solution

$Let\ n = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Longrightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = 0$

$3p + q + 3r = 0$ …………….. ….(i)

$\begin{pmatrix} p \\ q \\ r \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = 0$

$4p + 2q + 5r = 0$ ……………… (ii)

From Eqn (i);

$q = -3p – 3r$

$4p + 2(-3p – 3r) + 5r = 0$

$4p – 6p – 6r + 5r = 0$

$-2p – r = 0$

$r = -2p$

$q = -3p – 3(-2p)$

$q = 3p$

$$\begin{pmatrix}p\\q\\r\end{pmatrix} = \begin{pmatrix}p\\3p\\-2p\end{pmatrix} = p\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$n = \begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$n.\boldsymbol{r} = n.\boldsymbol{a}$$

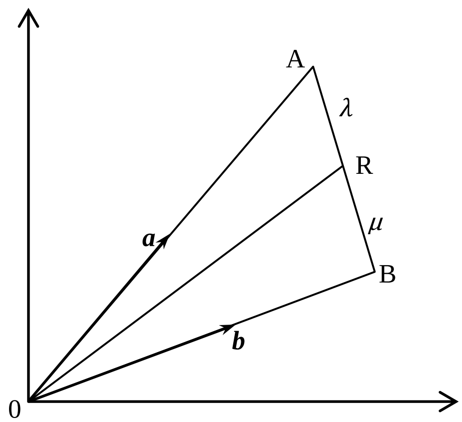
$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}1\\3\\-2\end{pmatrix}.\begin{pmatrix}-2\\5\\-11\end{pmatrix}$$

$x + 3y - 2z = -2 + 15 + 22$

$x + 3y - 2z = 35$

INTERNAL AND EXTERNAL DIVISIONS

Let A and B be points in space with position vectors A and B.



Let R be a point on a line segment AB dividing AB internally in the ratio of $\lambda : \mu$

$$\mathbf{OR = OA + AR}$$

$$OR = \boldsymbol{a} + \frac{\mu}{\lambda + \mu}AB$$

$$= \boldsymbol{a} + \frac{\boldsymbol{\lambda}}{\lambda + \mu}(\boldsymbol{b} - \boldsymbol{a})$$

$$OR = \frac{a\lambda + a\mu + b\lambda - a\lambda}{\lambda + \mu}$$

$$OR = \frac{a\mu + b\lambda}{\lambda + \mu}$$

Example I

Given that; $OP = \begin{pmatrix}4\\-3\\5\end{pmatrix}, OQ = \begin{pmatrix}1\\0\\2\end{pmatrix}$. Find the coordinates of R such that $PR : RQ = 1:2$

$$\boldsymbol{r} = \frac{\boldsymbol{a}\mu + \boldsymbol{b}\lambda}{\lambda + \mu}$$

$$OR = 2\begin{pmatrix}4\\-3\\5\end{pmatrix} + 1\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$OR = \frac{\begin{pmatrix}9\\-6\\12\end{pmatrix}}{3}$$

$$OR = \frac{1}{3}\begin{pmatrix}0\\-6\\12\end{pmatrix}$$

$$R = (3, -2, 4)$$

Example II

The points A $\begin{pmatrix}2\\-1\\6\end{pmatrix}$ and $B\begin{pmatrix}7\\6\\1\end{pmatrix}$ form a line segment which is divided externally in the ratio of 4:-1. Find the coordinates of T

$$(OT) = \frac{-1\begin{pmatrix}2\\-1\\6\end{pmatrix} + 4\begin{pmatrix}7\\6\\1\end{pmatrix}}{-1 + 4}$$

$$OT = \frac{\begin{pmatrix}-2+28\\1+24\\-6+4\end{pmatrix}}{3}$$

$$= \left(\frac{1}{3}\right)\begin{pmatrix}26\\25\\-2\end{pmatrix}$$

$$OT = \left(\frac{26}{3}, \frac{25}{3}, -\frac{2}{3}\right)$$

Example III

Find the position vectors $\begin{pmatrix}3\\-2\\5\end{pmatrix}$ and $\begin{pmatrix}9\\1\\-1\end{pmatrix}$, Find the position vectors of C which divides AB externally in the ratio of 5:-3

Solution:

$$\frac{-3\begin{pmatrix}3\\-2\\5\end{pmatrix} + 5\begin{pmatrix}9\\1\\-1\end{pmatrix}}{5 + \ -3}$$

$$\frac{\begin{pmatrix}-9\\6\\-15\end{pmatrix} + \begin{pmatrix}45\\5\\-5\end{pmatrix}}{2}$$

$$\frac{\begin{pmatrix}-9+45\\6+5\\-15-5\end{pmatrix}}{2}$$

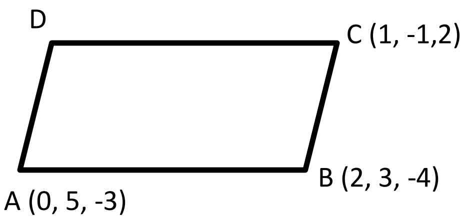
$\frac{1}{2}\begin{pmatrix}36\\11\\-20\end{pmatrix}$

$OC = \begin{pmatrix}18\\11/2\\10\end{pmatrix}$

$C = \left(18, \frac{11}{2}, -10\right)$

Example IV

Given that A(0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle or parallelogram.



$AB = DC$

$(OB - OA) = (OC - OD)$

$OD = OC + OA - OB$

$OD = \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}0\\5\\-3\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix}$

$OD = \begin{pmatrix}1\\4\\-1\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix}$

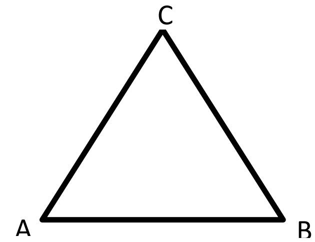
$OD = \begin{pmatrix}-1\\1\\3\end{pmatrix}$

$D = (-1, 1, 3)$

Proving that three points are vertices of a triangle

Give a triangle ABC with vertices

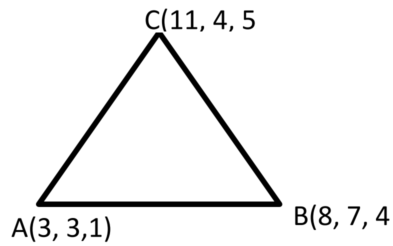
$A = (x_1 y_1 z_1)$ B (x_2, y_2, z_2) C (x_3, y_3, z_3)



$$\boldsymbol{AB + BC + CA = 0}$$
$$\boldsymbol{OB - OA + OC - OB + OA - OC = 0}$$

Example

Show that $3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}, 8\boldsymbol{i} + \boldsymbol{7j} + 4\boldsymbol{k}$ **and** $\boldsymbol{11i} + \boldsymbol{4j} + \boldsymbol{5k}$ are vertices of a triangle



$AB + BC + CA = 0$

$OB - OA + OC - OB + OA - OC$

$= \begin{pmatrix}8\\7\\4\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} + \begin{pmatrix}11\\4\\5\end{pmatrix} - \begin{pmatrix}8\\7\\4\end{pmatrix} + \begin{pmatrix}3\\3\\1\end{pmatrix} - \begin{pmatrix}11\\4\\5\end{pmatrix}$

$= \begin{pmatrix}5\\4\\3\end{pmatrix}\begin{pmatrix}3\\-3\\1\end{pmatrix} + \begin{pmatrix}-8\\-1\\-4\end{pmatrix}$

$\begin{pmatrix}8\\1\\4\end{pmatrix} + \begin{pmatrix}-8\\-1\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = 0$

Length and the equation of the perpendicular drawn from the point

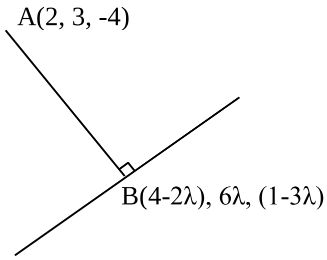
Example I

Find the equation and length of the perpendicular drawn from a point (2, 3, -4) to the line

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$

Solution

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$
$$\Rightarrow \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$$



$$r = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2\lambda \\ 6\lambda \\ -1 - 3\lambda \end{pmatrix}$$

$$AB. \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$AB = OB - OA$$

$$= \begin{pmatrix} 4 - 2\lambda - 2 \\ 6\lambda - 3 \\ 1 - 3\lambda - 4 \end{pmatrix} = \begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix}$$

$$\begin{pmatrix} 2 - 2\lambda \\ 6\lambda - 3 \\ 5 - 3\lambda \end{pmatrix} \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix} = 0$$

$$(-2(2 - 2\lambda) + 6(6\lambda - 3) - 3(5 - 3\lambda) = 0$$

$$-4 + 4\lambda + 36\lambda - 18 - 15 + 9\lambda = 0$$

$$36\lambda + 9\lambda + 4\lambda - 18 - 15 - 4 = 0$$

$$49\lambda = 37$$

$$\lambda = \frac{37}{49}$$

$$AB = \begin{pmatrix} 2 - 2\left(\frac{37}{49}\right) \\ 6\left(\frac{37}{49}\right) - 3 \\ 5 - 3\left(\frac{37}{49}\right) \end{pmatrix}$$

$$AB = \begin{pmatrix} 24/49 \\ 75/49 \\ 134/49 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} \frac{24}{49} \\ \frac{75}{49} \\ \frac{134}{49} \end{pmatrix}$$

Equation of the perpendicular

$$\frac{x-2}{72/49} = \frac{y-3}{-69/49} = \frac{z-4}{186/49}$$

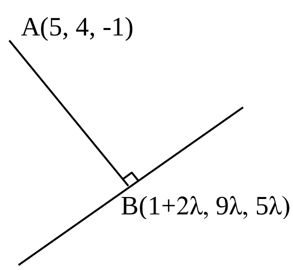
Length of the perpendicular AB

$$AB = \sqrt{\left(\frac{24}{49}\right)^2 + \left(\frac{75}{49}\right)^2 + \left(\frac{134}{49}\right)^2}$$

AB = 3.1719 units

Find the length and equation of the perpendicular drawn from a point (5, 4, -1) to the line; $\boldsymbol{r} = \boldsymbol{i} + \lambda(2\boldsymbol{i} + 9\boldsymbol{j} + 5\boldsymbol{k})$

Solution



$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 9\lambda \\ 5\lambda \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 + 2\lambda - 5 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix} = \begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix}$$

$$AB.d = 0$$

$$d = \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 2\lambda - 4 \\ 9\lambda - 4 \\ 5\lambda + 1 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \\ 5 \end{pmatrix} = 0$$

$$(2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0$$

$4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$
$81\lambda + 25\lambda + 4\lambda - 8 + 5 - 36 = 0$
$110\lambda = 39$
$\lambda = \frac{39}{110}$

$$AB = \begin{pmatrix} 2\left(\frac{39}{110}\right) - 4 \\ 9\left(\frac{39}{110}\right) - 4 \\ 5\left(\frac{39}{110}\right) + 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{-305}{110} \end{pmatrix}$$

$$|AB| = \sqrt{\left(\frac{-362}{110}\right)^2 \left(\frac{-89}{110}\right)^2 \left(\frac{-305}{110}\right)^2}$$

$|AB| = 4.379$ units

Equation of the perpendicular bisector is

$$r = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} \frac{-362}{110} \\ \frac{-89}{110} \\ \frac{305}{110} \end{pmatrix}$$

$$\frac{x-5}{{-362}/{110}} = \frac{y-4}{{-89}/{110}} = \frac{z+1}{{305}/{110}} = \mu$$

Shortest Distance between Parallel Planes

Example I
Find the perpendicular distance between two parallel planes;

$2x + 5y - 14z = 30$
$2x + 5y - 14z = -15$

Solution

$r.\hat{n} = d_1$

Plane 1

$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = \frac{30}{15}$$

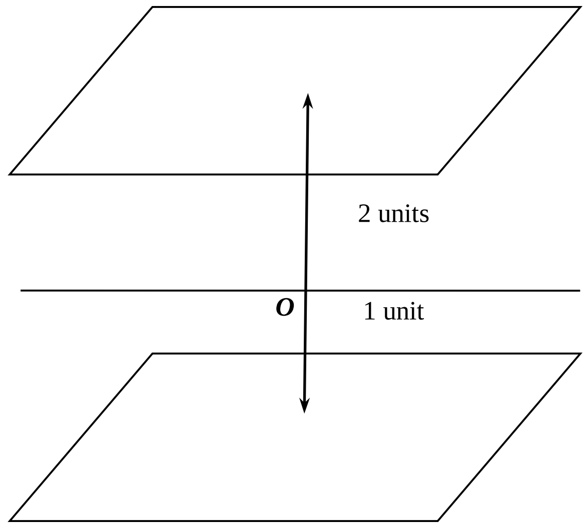
$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = 2$$

$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = 2$$

Plane 2

$r.2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k} = -15$

$$r.\left(\frac{2\boldsymbol{i} + 5\boldsymbol{j} - 14\boldsymbol{k}}{15}\right) = -1$$



Example II
Find the perpendicular distance between two parallel planes;
$x + 2y - z = -4$ *and* $x + 2y - z = 3$
$r.\hat{n} = d_1$
For plane 1

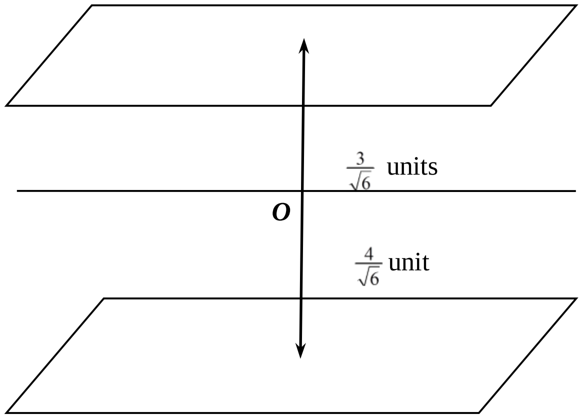
$r.(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -4$

$$r.\frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{6}} = \frac{-4}{\sqrt{6}}$$

For plane 2

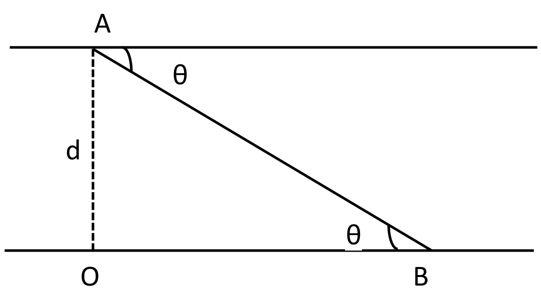
$r.(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$

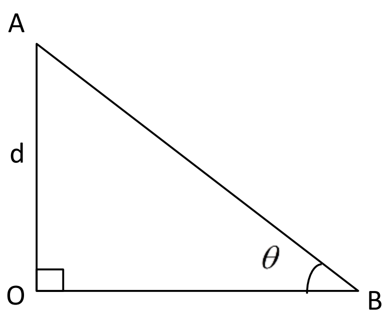
0 $$r.\frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$



$$= \frac{3}{\sqrt{6}} + \frac{4}{\sqrt{6}} = \frac{7}{\sqrt{6}} units$$

Shortest distance between two parallel lines





Distance between a point A and line B

$$d = AB\sin\theta$$

Example I

Find the shortest distance between the following pairs of parallel lines

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$$

and

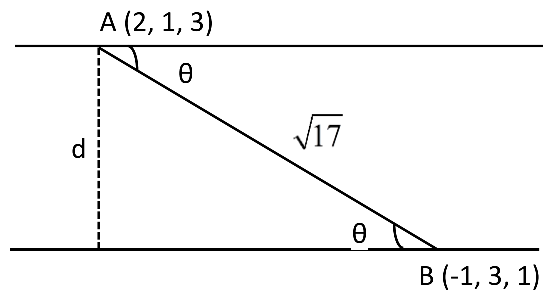
$$\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$$

$$AB = \sqrt{(2+1)^2 + (1-3)^2 + (3-1)^2}$$

$$AB = \sqrt{17}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$$



$$\cos\theta = \frac{AB.d}{|AB|.d}$$

$$\cos\theta = \frac{\begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{17}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{100}}\right)$$

$$\theta = 26.8°$$

$$\sin 26.8° = \frac{d}{\sqrt{17}}$$

$$d = 1.859\ units$$

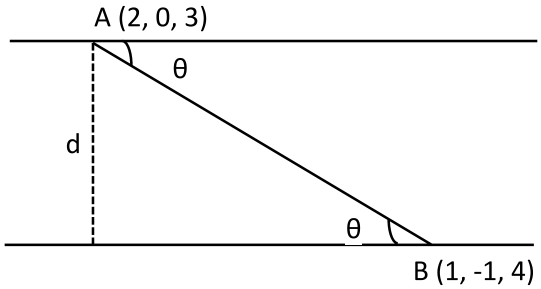
Example II

Find the distance between the following pairs of parallel lines

$$r = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Solution



$$AB=\sqrt{(2-1)^2+(0+1)^2+(3-4)^2}$$
$$AB=\sqrt{1+1+1}$$
$$AB=\sqrt{3}$$
$$\cos\theta=\frac{2}{\sqrt{18}}$$
$$\theta=\cos^{-1}\left(\frac{2}{\sqrt{18}}\right)$$
$$\theta=61.9°$$
$$\sin\theta=\frac{d}{\sqrt{3}}$$
$$\sin 61.9°=\frac{d}{\sqrt{3}}$$
$$d=\sqrt{3}\sin 61.9°$$
$$d=1.52789\ units$$

SKEW LINES

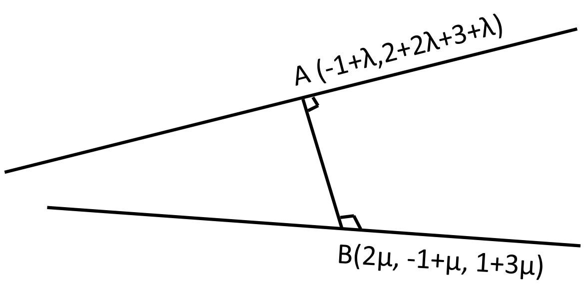
These are lines which are neither parallel nor perpendicular

Shortest distance between two skew lines

Example I

Find the shortest distance between the following skew lines

$$\boldsymbol{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix} \text{ and } \boldsymbol{r}=\begin{pmatrix}0\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$$



$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

$$\begin{pmatrix}2\mu-(1+\lambda)\\-1+\mu-(2+2\lambda)\\1+3\mu-(3+\lambda)\end{pmatrix}-\begin{pmatrix}2\mu-\lambda+1\\\mu-2\lambda-3\\3\mu-\lambda-2\end{pmatrix}$$

$$\begin{pmatrix}2\mu-\lambda+1\\\mu-2\lambda-3\\3\mu-\lambda-2\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=0$$

$$2\mu-\lambda+1+2\mu-4\lambda-6+3\mu-\lambda-2=0$$
$$7\mu-6\lambda=7\text{.......................}(1)$$

$$\begin{pmatrix}2\mu-\lambda+1\\\mu-2\lambda-3\\3\mu-\lambda-2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}=0$$

$$4\mu-2\lambda+2+\mu-2\lambda-3+9\mu-3\lambda-6=0$$
$$14\mu-7\lambda-7=0$$
$$14\mu-7\lambda=7\text{........................}(2)$$
$$\mu=\frac{-1}{5},\lambda=-\frac{7}{5}$$

$$AB=\begin{pmatrix}2\\-0.4\\-1.2\end{pmatrix}$$

$$AB=\sqrt{2^2+(-0.4)^2+(-1.2)^2}=2.3664units$$

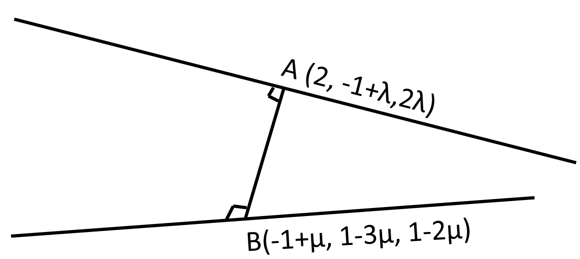
Example II

Find the shortest distance between the following pairs of skew lines

$$\frac{x-2}{0}=\frac{y+1}{1}=\frac{z}{2} \text{ and } \frac{x+1}{1}=\frac{y-1}{-3}=\frac{z-1}{-2}$$

Solution

$$\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\2\end{pmatrix},\ \mathbf{r}=\begin{pmatrix}-1\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\-2\end{pmatrix}$$



$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}\begin{pmatrix}(-1+\mu)-2\\(1-3\mu)-(-1+\lambda)\\(1-2\mu)-2\lambda\end{pmatrix}=\begin{pmatrix}\mu-3\\-3\mu-\lambda+2\\1-2\mu-2\lambda\end{pmatrix}$$

$$\begin{pmatrix}\mu-3\\-3\mu-\lambda+2\\1-2\mu-2\lambda\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}=0$$

$$-3\mu-\lambda+2+2-4\mu-4\lambda=0$$

$$-7\mu-5\lambda-4=0$$

$$7\mu+5\lambda-4=0$$

$$7\mu+5\lambda=4\text{..........................(1)}$$

$$\begin{pmatrix}\mu-3\\-3\mu-\lambda+2\\1-2\mu-2\lambda\end{pmatrix}\begin{pmatrix}1\\-3\\-2\end{pmatrix}=0$$

$$\mu-3+9\mu+3\lambda-6-2+4\mu+4\lambda$$

$$14\mu+7\lambda-11=0$$

$$14\mu+7\lambda=11\text{......................(2)}$$

$$AB=\begin{pmatrix}-12/7\\-6/7\\3/7\end{pmatrix}$$

$$\mu=\frac{9}{7},\lambda=-1$$

$$AB=\sqrt{\left(\frac{-12}{7}^2\right)+\left(\frac{-6}{7}\right)^2+\left(\frac{3}{7}\right)^2}$$

$$AB=\sqrt{\frac{144}{49}+\frac{36}{49}+\frac{9}{49}}$$

$$AB=\frac{3\sqrt{21}}{7}\,units$$

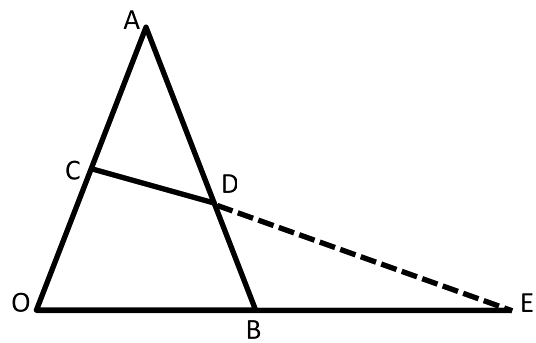
Vector Geometry

Example I

Triangle OAB has OA=**a,** OB=**b**. C is a point on OA such that OC= $^2/_3$ **a**. D is a mid point of AB when CD is produced, it meets OB at E such that DE = nCD and BE=**k**b. Express BE, DE in terms of;

a) **n, a** and **b**

b) **k, b** and **a**. Hence find the values of **n** and **k.**



$$\overrightarrow{DE}=n\overrightarrow{CD}$$

$$\overrightarrow{DE}=n\left[\overrightarrow{CA}+\overrightarrow{AD}\right]$$

$$\overrightarrow{DE}=n\left[\frac{1}{3}\mathbf{a}+\overrightarrow{AD}\right]$$

$$\overrightarrow{DE}=n\left[\frac{1}{3}\mathbf{a}+\frac{1}{2}\overrightarrow{AB}\right]$$

$$\overrightarrow{DE}=\frac{1}{3}n\mathbf{a}+\frac{1}{2}n\mathbf{b}-\frac{1}{2}n\mathbf{a}$$

$$\overrightarrow{DE}=\frac{-1}{6}n\mathbf{a}+\frac{1}{2}n\mathbf{b}\text{...................(1)}$$

$$\overrightarrow{DE}=\overrightarrow{DB}+\overrightarrow{BE}$$

$$\overrightarrow{DE}=\frac{1}{2}\overrightarrow{AB}+k\mathbf{b}$$

$$\overrightarrow{DE}=\frac{1}{2}(\mathbf{b}-\mathbf{a})+k\mathbf{b}$$

$$\overrightarrow{DE}=\frac{1}{2}\mathbf{b}-\frac{1}{2}\mathbf{a}+k\mathbf{b}$$

$$\overrightarrow{DE}=\left(\frac{1}{2}+k\right)\mathbf{b}-\frac{1}{2}\mathbf{a}$$

$$\frac{-1}{2}\mathbf{a}=-\frac{1}{6}n\mathbf{a}$$

$$\frac{1}{2}=\frac{1}{6}n$$

$$6=2n$$

$$n=3$$

$$\left(\frac{1}{2}+k\right)\mathbf{b}=\frac{1}{2}n\mathbf{b}$$

$$\frac{1}{2}+k=\frac{1}{2}\times 3$$

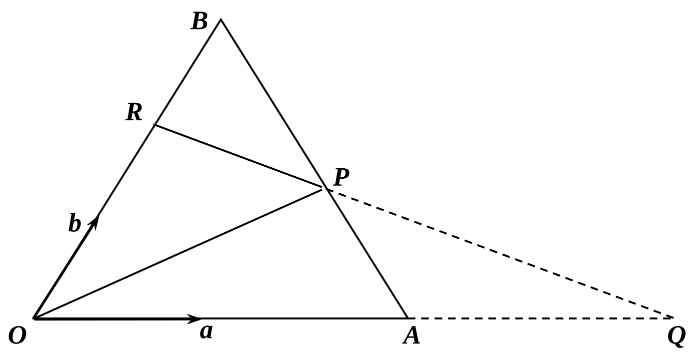
$$k=\frac{3}{2}-\frac{1}{2}=1$$

Example II

Given that OA is **a** and OB=**b** point R is on OB such that OR:RB=4:1. Point P is on AB such that BP:PA=2:3. When RP and OA are both produced, they meet at Q. Find OR and OP in terms of **a** and **b**

ii) OQ in terms of **a**

Solution



$$\overrightarrow{OR}=\frac{4}{5}OB \qquad \Rightarrow \overrightarrow{OR}=\frac{4}{5}\mathbf{b}$$

$$\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{BP}$$

$$\overrightarrow{OP}=\mathbf{b}+\frac{2}{5}\overrightarrow{BA}$$

$$\overrightarrow{OP}=\mathbf{b}+\frac{2}{5}(a-b)$$

$$\overrightarrow{OP}=\frac{1}{5}(3\mathbf{b}+2\mathbf{a})$$

$$\overrightarrow{OQ}=\lambda\overrightarrow{OA}=\lambda\mathbf{a}$$

$$\overrightarrow{OQ}=\overrightarrow{OR}+\overrightarrow{RQ}$$

$$\overrightarrow{OQ}=\frac{4}{5}\mathbf{b}+\mu\overrightarrow{RP}$$

$$\overrightarrow{OQ}=\frac{4}{5}\mathbf{b}+\mu\left(\frac{-4}{5}\mathbf{b}+\frac{1}{5}(2\mathbf{a}+3\mathbf{b})\right)$$

$$\overrightarrow{OQ}=\left(\frac{4}{5}-\frac{1}{5}\mu\right)\mathbf{b}+\frac{2}{5}\mu\mathbf{a}$$

$$\frac{4}{5}-\frac{1}{5}\mu=0$$

$$\mu=4$$

$$\lambda=\frac{2}{5}\mu$$

$$\lambda=\frac{8}{5}$$

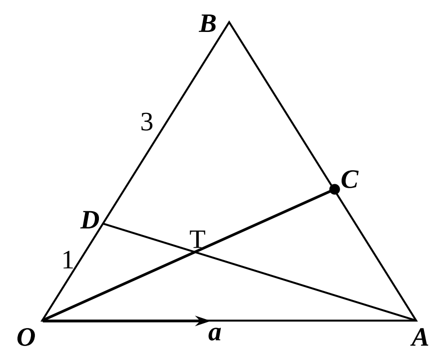
$$\text{OQ} = \frac{8}{5}\mathbf{a}$$

Example III

O, A and B are non collinear points OA = a, OB = b, C is midpoint of AB, D is a point on OB such that $\mathbf{OD} = \frac{1}{4}\mathbf{OB}$.

T is a point of intersection of OC and AD. Find the vector OT in terms of *a* and *b*.

Solution



OT = λOC

OC = OB + BC

$= \mathbf{b} + \frac{1}{2}\text{BA}$

$= \mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b})$

OC = $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

$\mathbf{OT}=\lambda\left(\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$

$\mathbf{OT}=\frac{1}{2}\lambda\mathbf{a}+\frac{1}{2}\lambda\mathbf{b}$..........................(i)

OT = OA + AT

= a + μ **AD**

AD = AO + OD

$= \mathbf{a} + \frac{1}{4}\mathbf{b}$

OT = a + μ **(a +** $\frac{1}{4}$**b)**

OT = a – μ **a +** $\frac{1}{4}$ μ **b**

$$\mathbf{OT} = (1 - \mu)\mathbf{a} + \frac{1}{4}\mu\,\mathbf{b} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

Equating components of vectors **a** and **b** in Eqns (i) and (ii);

$$\frac{1}{2}\lambda = 1 - \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$\frac{1}{2}\lambda = \frac{1}{4}\mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

From Eqn (iv);

$$2\lambda = \mu$$

$$\Rightarrow \frac{\lambda}{2} = 1 - 2\mu$$

$$\frac{5\lambda}{2} = 1$$

$$\lambda = \frac{2}{5}$$

$$\mu = \frac{4}{5}$$

$$\mathbf{OT} = \frac{2}{5}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$$

$$\mathbf{OT} = \frac{1}{5}(\mathbf{a} + \mathbf{b})$$

Revision Exercise

1. In a triangle *ABC*, the altitudes from B and C meet the opposite sides at E and F respectively. BE and CF intersect at O. Taking O as the origin, use the dot product to prove that *AO* is perpendicular to BC
   (b) Find the point of intersection of the line $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ with the plane $3x + 4y + 2z - 25 = 0$
   (c) Find the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane $4x + 3y + 1 = 0$
2. (a) Show that the equation of the plane through points A with position vector 2i + 2k perpendicular to the vector i + 3j – 2k is $x + 3y - 2z + 10 = 0$
   (b) (i) Show that the vector 2i – 5j + 3.5k is perpendicular to the line **r** = 2i – j + λ(4i + 3j + 2k)
   (ii) Calculate the angle between the vector 3i – 2j + k and the line in (b)(i) above.
3. A point P has coordinates (1, -2, 3) and a certain plane has the equation $x + 2y + 2z = 8$. The line through P parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = \frac{z+1}{-2}$ meets the plane at a point Q.
4. (a) The line through A(1, -2, 2) and perpendicular to the plane $4x - y + 2z + 12 = 0$ meets the plane in point B. Find the coordinates of B.
   (b) Given that the vectors a**i** – 2**j** + **k** and 2a**i** + a**j** – 4**i** are perpendicular, find the values of *a*.
5. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the vector **r** = 4i + 5j + k.
6. (a) The vertices of a triangle are P(2, -1, 5), Q(7, 1, -3) and R(13, -2, 0). Show that $\angle PQR = 90°$. Find the coordinates of S if PQRS is a rectangle.
   (b) Find the equation of the line through A(2, 2, 5) and B(1, 2, 3)
   (c) If the line in (b) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at P, find the:
   (i) coordinates of P,
   (ii) angle between the two lines
7. The position vector of points P and Q are 2i – 3j and 3i – 7j + 12k respectively. Determine the length of PQ. PQ meets the plane $4x + 5y - 2z = 5$ at point S. Find:
   (a) the coordinates of *S*,
   (b) the angle between PQ and the plane.
8. (a) Find the angle between the line **r** = 3k + λ(7i – j + 4k) and the plane **r**· (2i – 5j – 2k) = 8
   (b) Show that the lines with vector equations
   $\mathbf{r}_1 = (1 + 4\lambda)i + (1 - \lambda)j + (2\lambda)k$, and
   $\mathbf{r}_2 = (5 + 3\mu)i + (2\mu)j + (2 - 5\mu)k$.
   intersect at right angles and give the position vector of the point of intersection.
9. Find the equation of the line with directrix vector **d** which passes through the point with position vector **a** given that
   (a) $\mathbf{a} = i + 2j - k$, $\mathbf{d} = 3i - k$
   (b) $\mathbf{a} = 4i - 3k$, $\mathbf{d} = i - 3j + 3k$
10. Find the vector equation of the line which passes through the points with (a) position vectors $3i - 3j + k$ and $-2j + j + k$.
    (a) position vector $i + 4j$ and $3i - j + 2k$,
    (b) coordinates (0, 6, -6) and (5, -7, 2)
    (c) coordinates (0, 0, 0) and (5, -2, 3)
11. Write down in parametric form the vector equations of the planes through the given points parallel to the given pairs of vectors.
    (a) (1, -2, 0); i + 3j and –j + 2k
    (b) the origin; 2i – j and –i + 2j – 7k
    (c) (3, 1, -1); j and i + j + k.

12. Find a vector equation for the plane passing through the points with position vectors $2k$, $i - 3j + k$ and $5i + 2j$.
13. Find the vector equation of the plane through the points A(1, 0, -2) and B(3, -1, 1) which is parallel to the line with vector equation $\mathbf{r} = 3i + (2\lambda - 1)j + (5 - \lambda)k$. Hence find the coordinates of the point of intersection of the plane and the line $\mathbf{r} = \mu i + (5 - \mu)j + 2\mu - 7)k$.
14. Find a vector equation for the line joining the points
    (a) (2, 6) and (5, 2)
    (b) (-1, 2, -3) and (6, 3, 0).
15. (a) Points A and B have coordinates (4, 1) and (2, -5) respectively. Find a vector equation for the line which passes through A and perpendicular to the line AB.
    (b) Points P and Q have coordinates (3, 5) and (-3, -7) respectively. Find a vector equation for the line which passes through the point P and which is perpendicular to the line PQ
16. Find a vector equation for the perpendicular bisector of the points:
    (a) (6, 3) and (2, -5)
    (b) (7, -1) and (3, -3)
17. Points P, Q and R have position vectors 4i – 4j, 2i + 2j, and 8i + 6j respectively.
    (a) Find a vector equation for the line L_1 which is the perpendicular bisector to the points P and Q
    (b) Find a vector equation for the line L_2 which is the perpendicular bisector to the points A and R.
    (c) Hence find the position vector of the point where L_1 and L_2 meet.
18. Two lines L_1 and L_2 have equations

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \text{ and } L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

    (a) Show that L_1 and L_2 are concurrent (meet at a common point) and find the position vector of their point of intersection.
    (b) Find the angle between L_1 and L_2.
19. Points P, Q, and R have coordinates (-1, 1), (4, 6) and (7, 3) respectively.
    (a) Show that the perpendicular distance from the point R to the point PQ is $3\sqrt{2}$.
    (b) Deduce that the area of the triangle PQR is 15 sq.units.
20. Points A, B and C have position vectors –i + 3j + 9k, 5i + 6j – 4k and 4i + 7j + 5k respectively. P is the point on AB such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$. Find:
    (a) $\overrightarrow{AB}$
    (b) $\overrightarrow{CP}$
    (c) Find the perpendicular distance from the point C to the line AB.
21. Two lines L_1 and L_2 have vector equations
    $\mathbf{r}_1 = (2 - 3\lambda)i + (1 + \lambda)j + 4\lambda k$
    $\mathbf{r}_2 = (-1 + 3\lambda)i + 3j + (4 - \lambda)k$ respectively. Find:
    (a) the position vector of their common point of intersection.
    (b) the angle between the lines.
22. Find the equation of the plane containing points P(1, 1, 1), Q(1, 2, 0) and (-1, 2, 1).
23. Find the equation of the plane containing point (4, -2, 3) and parallel to the plane $3x - 7z = 12$
24. Show that the point with position vector 7i – 5j – 4k lies in the plane $r = 4i + 3j + 2k + \lambda(i - j - k) + \mu(2i + 3j + k)$. Find the point at which the line $x = y - 1 = 2z$ intersects the plane $4x - y + 3z = 8$.
25. Find the parametric equations for the line through the point (0, 1, 2) that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.
26. Find the distance between the parallel planes $z = x + 2y + 1$ and $3x + 6y - 3z = 4$
27. Two planes are given by the parametric equations
    $x = r + 3$ and $x = 1 + r + s$
    $y = 3s$ and $y = 2 + r$
    $z = 2r$ and $z = -3 + 5$
    Find the Cartesian equation of the intersection point.
28. The equation of a plane P is given by $r \cdot \begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} = 33$, where r is the position vector of P. find the perpendicular distance from the plane to the origin.
29. The line through point P(1, -2, 3) and parallel to the line $\frac{x}{3} + \frac{y+1}{-1} = z + 1$ meets the plane $x + 2y + 27\ 8$ at Q. find the coordinates of Q.
30. (a) Find the angle between the plane $x + 4y - z = 72$ and the line $\mathbf{r} = 9i + 6j + 8k$.

(b) obtain the equation of the plane that passes through (1, -2, 2) and perpendicular to the line $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{1}$

(c) Find the parametric equations of the line of intersection of the plane $x + y + z = 4$ and $x - y + 2z + 2 = 0$

31. Find the point of intersection of the three planes $2x - y + 3z = 4$, $3x - 2y + 6z = 3$ and $7x - 4y + 5z = 11$.

32. Find the Cartesian equation of the plane with parametric vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

33. Find the Cartesian equation of the plane containing the point with position vector $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and parallel to the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

34. Find the Cartesian equation of the plane containing the points with position vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$.

35. Find the perpendicular distance from the plane $\mathbf{r}.(2i - 14j + 5k) = 10$ to the origin.

36. Find the position vector of the point where the line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ meets the plane $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = 15$.

37. Two lines have vector equations $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.

38. The position vector of points P and Q are 3i - j + 2k and 2i + 2j + 3k, respectively. Find the acute angle between PQ and the line $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$.

(b) Find the point of intersection of the line $x - 2 = 2y + 1 = 3 - z$ and the plane $x + 2y + z = 3$.

(c) Find the equation of the plane through the origin parallel to the lines $\mathbf{r} = 3i + 3j - k + s(i - j - 2k)$ and $\mathbf{r} = 4i - 5j - 8k + t(3i + 7j - 6k)$

39. (a) The points A and B have position vectors $\mathbf{a} = 2i - j + 6k$ and $\mathbf{b} = 7i - 6j + k$ respectively. Find the coordinates of a point P which divides the vector AB in the ratio:

(i) 4:1

(ii) 1:4

40. (b) Find the Cartesian equation of the plane through the origin parallel to the lines $x - 3 = 3 - y = \frac{z+1}{-2}$ and $\frac{x-4}{3} = \frac{y+5}{7} = \frac{x+8}{-6}$

(c) Find the angle between the line $1 - x = \frac{y-3}{2} = \frac{4-z}{4}$ and the plane $2x - 3y - 2z + 5 = 0$.

41. (a) Determine the unit vector perpendicular to the plane containing the points A(0, 2, -4), B(2, 0, 2) and C(-8, 4, 0).

(b) Find the equation of the plane in (a) above

(c) Show that the point T(5, -4, 3) lies on the plane in (a) above.

(d) Write down the equation in the form $\mathbf{r} = a + \lambda b$ of the perpendicular through the point P(3, 4, 2) to the plane in (a) above.

(e) If the perpendicular meets the plane in (a) above at N, determine vector NP.