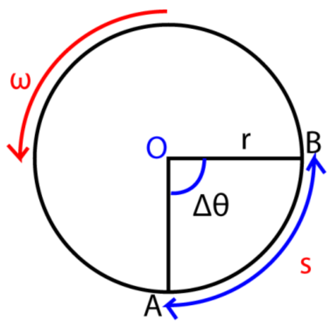
## Circular motion

This is the motion of an object moving in a circular path with a uniform speed around a fixed point O. consider a body moving from A to B in a small time Δt such that the radius r, sweeps through a small angle Δθ in radians




Distance, AB = s = rΔθ

But speed, v = $\frac{distance}{time} = r\frac{\Delta\theta}{\Delta t}$

As Δt → 0, $\frac{\Delta\theta}{\Delta t} \rightarrow \frac{d\theta}{dt}$

v = $r\frac{d\theta}{dt}$

Since, $\frac{d\theta}{dt}$ = ω

v = rω

Terminology in circular motion

**Angular velocity, ω**

This is the rate of the angle for an object moving in a circular path about in a circular path about the center. S.I units are $rads^{-1}$.

**Period, T**

Time taken to make one complete revolution

Time = $\frac{Distance}{speed}$

$= \frac{2\pi r}{v}$

$= \frac{2\pi r}{r\omega}$

$= \frac{2\pi}{\omega}$

**Frequency, f**
This is the number of revolutions made in one second. The S.I units are hertz (Hz)

$F = \frac{1}{T}$

$= \frac{\omega}{2\pi}$

Centripetal acceleration (a)
This is the rate of change of velocity for a body moving in a circular path and it is directed towards the center of that circular path.

$a = \frac{v^2}{r} = r\omega^2$

## Derivation of a = $\frac{v^2}{r}$ *or* $r\omega^2$

Consider an object moving with a constant speed, v, round a circle of radius, r.

In figure (i) below; at A, its velocity $v_A$ is in direction of the tangent AC, a short time dt later at B, its velocity $v_B$ is in the direction of tangent BD

Since their directions are different, the velocity $v_B$ is different from the velocity $v_A$ although their magnitude are both equal to v.

Thus a velocity change or acceleration has occurred from A to B

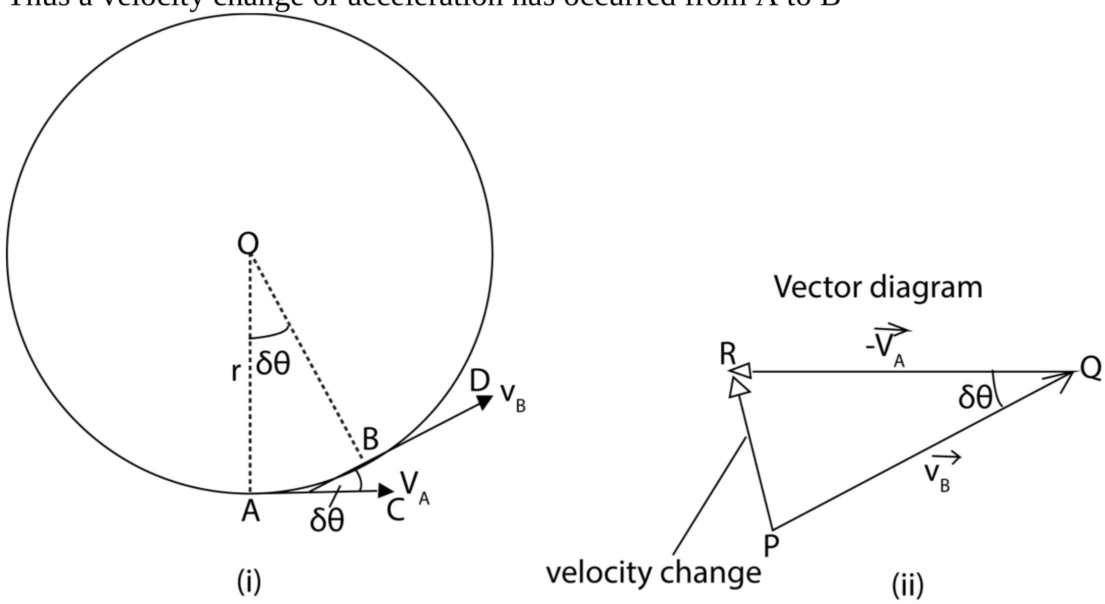



Fig. 2: Acceleration in circle

The velocity change from a to B = $v_B - v_A$ or $v_B + (-v_A)$.

In figure 2(ii) above, PQ represents $v_B$ in magnitude (v) and direction BD; QR represents $-v_A$ in magnitude (v) and direction (CA).

Velocity change = $v_B + (-v_A)$ = PR

When δt is small, the angle AOB or δθ is small;
Also angle PQR equal to δθ is small

PR or acceleration then points toward O, the center of the circle.

a = $\frac{velocity\ change}{time} = \frac{PR}{\delta t} = \frac{v\delta\theta}{\delta t}$

but $\frac{\delta\theta}{\delta t}$ =ω and v = rω

a = rω x ω = rω$^2$

thus an object moving in a circle of radius r with a constant speed v has a constant acceleration towards the center equal to $\frac{v^2}{r} = r\omega^2$

**Centripetal force**

This is the force which keeps the body moving in a circular path and it is directed towards the center of the circular path.

F = ma = m$\frac{v^2}{r} = mr\omega^2$

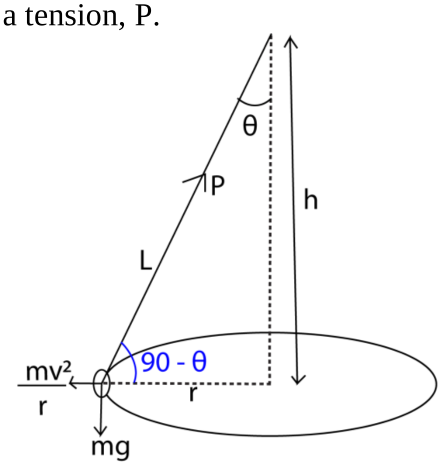
**Centrifugal force**

This is the force acting on a body moving in a circular path.

## Examples of circular motion

1. A conical circular pendulum

Consider a mass , m, attached to a string of length L moving around a horizontal circle of radius, R, at a constant speed, v, and the string make an angle θ with the vertical and have a tension, P.




Resolving vertically

Pcosθ = mg …………………………………………….. (i)

Resolving horizontally

Psinθ = m$\frac{v^2}{r}$ ……………………………………………… (ii)

Eqn (i) and Eqn (ii)

tan θ =$\frac{v^2}{rg}$

From the diagram

sin θ = $\frac{r}{L}$, r = Lsinθ

$\cos\theta = \frac{h}{L}$, r = Lcos θ

Pcos θ = mg

⇨ $P \cdot \frac{h}{L} = mg$

$P = \frac{mgL}{h}$ …………………………….. (iii)

$P\sin\theta = m\frac{v^2}{r}$

$P \cdot \frac{r}{L} = m\frac{v^2}{r}$

$P = \frac{mv^2L}{r^2}$

But v = rω

$P = \frac{m(r\omega)^2L}{r^2} = mL\omega^2$ …………………….. (iv)

Eqn (iii) and (iv)

$mL\omega^2 = \frac{mgL}{h}$

$\omega^2 = \frac{g}{h}$

From $T = \frac{2\pi}{\omega}$

$\omega = \frac{2\omega}{T}$

$\frac{g}{h} = \left[\frac{2\pi}{T}\right]^2 = \frac{4\pi^2}{T^2}$

$T^2 = \frac{4\pi^2 h}{g}$

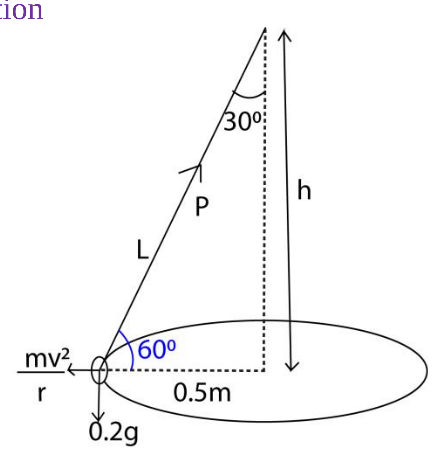
$T = \sqrt{\frac{2\pi^2 h}{g}} = 2\pi\sqrt{\frac{h}{g}}$

**Example 1**

A mass of 0.2kg is whirled in a horizontal circle of radius 0.5m by a string inclined at 300 to the vertical. Find

(i) The tension in the string
(ii) The speed of the mass in the horizontal plane
(iii) The length of the string
(iv) The angular speed

Solution




(i) Resolving vertically

Pcos θ = mg

$P = \frac{mg}{\cos\theta} = \frac{0.2 \ x \ 9.81}{\cos 30} = 2.2655N$

Tension P = 2.2655N

(ii) Resolving vertically

$P\sin\theta = m\frac{v^2}{r}$

$v^2 = \frac{rP\sin\theta}{m}$

$v = \sqrt{\frac{rP\sin\theta}{m}} = \sqrt{\frac{(0.5 \ x \ 2.2655 \ x \ \sin 30^0}{0.2}} = 1.6828ms^{-1}$

(iii) From $\sin\theta = \frac{r}{L}$

$L = \frac{0.5}{\sin 30} = 1m$

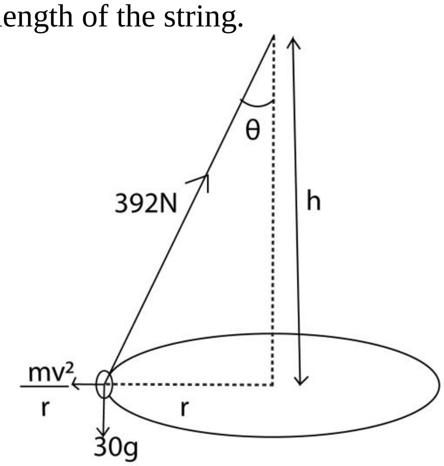
(iv) From $\omega = \frac{v}{r} = \frac{1.6828}{0.5} = 3.265rads^{-1}$

**Example 2**

A 30kg body is swirled in a horizontal circle as a conical pendulu by means of inelastic string that has a breaking strength of 392N. when the speed of the body is $8ms^{-1}$, the string broke. Calculate

(i) The angle the string made at that instant.

(ii) The length of the string.




(i) From $P\cos\theta = mg$

$\cos\theta = \frac{30 \ x \ 9.81}{392}$

$\theta = 41.34$

(ii) From $P\sin\theta = m\frac{v^2}{r}$

$r = \frac{30 \ x \ 8^2}{392 \sin 41.34^0} = 7.42$

But $\sin\theta = \frac{r}{L}$

$L = \frac{7.42}{\sin 41.34} = 11.23m$

**Example 3**

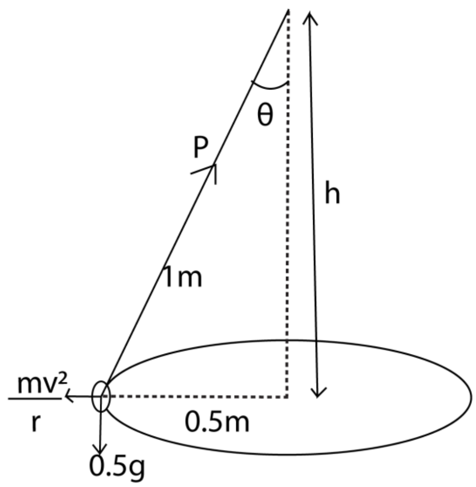
A steel ball of mass 0.5kg is suspeneded from a light inelastic string of length 1m, the ball is whirled in horizontal cirle of radis 0.5m. find

(i) Centripetal force and tension in the string

(ii) The angular speed of the ball

(iii) The angle between the string and the radius of the circle is thetension in string is 10N

Solution




(i) From sinθ = $\frac{r}{L}$

$\theta = \sin^{-1}\frac{0.5}{1} = 30^0$

From Pcos θ = mg

Tension, P = $\frac{mg}{\cos\theta} = \frac{0.5\ 9.81}{\cos 30}$= 5.664N

But, Psinθ = F

F = 5.664 sin 30= 2.832N

(ii) tanθ = $\frac{v^2}{rg}$

v = $\sqrt{rgtan\ \theta} = \sqrt{[0.5\ x\ 9.81\ x \tan 30^0]}$ = 1.683

But, ω =$\frac{v}{r} = \frac{1.683}{0.5} = 3.366rads^{-1}$

(iii) From Pcos θ = mg

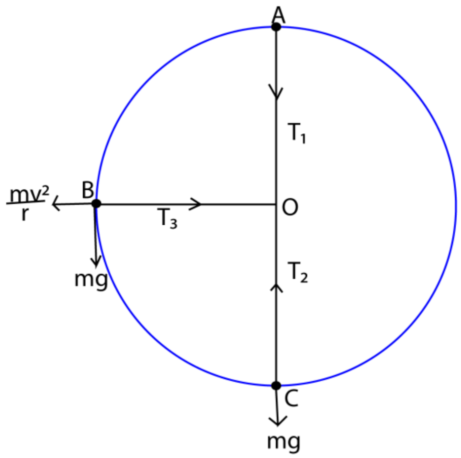
cos θ = $\frac{0.5x\ 9.81}{10}$

θ = 60.60

the required angle = 90 – 60.6 = 29.4$^0$

## Motion in verticle circle

Consider a body of mass, m, attached to a string of length, r, and whirled in a vertical circle at constant velecity, v.




At A, $m\frac{v^2}{r} = T_1 + mg$

$T_1 = m\frac{v^2}{r} - mg$

At B, $T_3 = m\frac{v^2}{r}$

At C, $T_4 = m\frac{v^2}{r} + mg$

From the above expressions, tension in the string is minimum aat the top of the circle and maximum at the bottom of the circle. So the string is most likely to break when the body is at the bottom of the circle.

## Example 4

A mass of 0.4kg is rotated by a string at a constant speed, v, in a vertical circle of radius 1m. If the minimum tension in a string is 3N. calculate

(i) The velocity
(ii) The maximum tension
(iii) Tension when the string is just horizontal

### Solution

Minimum tension, $T_1 = m\frac{v^2}{r} - mg$

$3 = \frac{0.4v^2}{1} - 0.4 \, x \, 9.81$

$v = 4.16ms^{-1}$

Maximum tension $T_3 = m\frac{v^2}{r} + mg$

$= \frac{0.4 \, x \, 4.16^2}{1} + 0.4 \, x \, 9.81$

$= 10.85N$

Tension when the string is horizontal $T = m\frac{v^2}{r} = \frac{0.4 \, x \, 4.16^2}{1} = 6.92N$

### Example 5

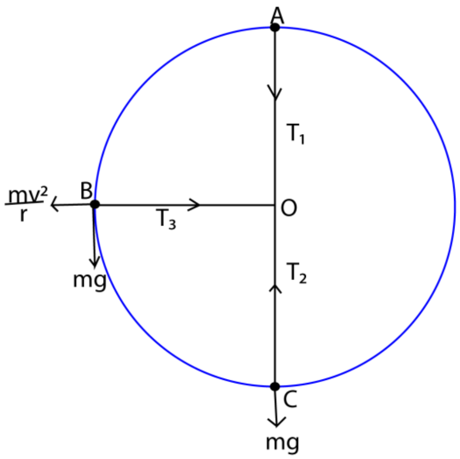
A particle of mass 5kg describes a complete vertical circle at the end of a light inextensible string of length 2m. given that the speed of the particle is $5ms^{-1}$ at the highest point. Find

(i) Speed at the lowest point

(ii) Tension in the string when it is horizontal

(iii) Magnitude of centripetal acceleration when the string is horizontal

Solution




(i) Mechanical energy at A = m$\frac{v^2}{r}$ + mg

r = 2m, m = 5kg, v = $5ms^{-1}$

Mechanical energy at A = $\frac{1}{2} x\ 5\ x\ 5^2 + 5\ x\ 9.81\ x\ 4$ = 258.7J

Mechanical energy at C = m$\frac{v^2}{r}$ + mg = $\frac{1}{2} x\ 5\ x\ v^2 + 5\ x\ 9.81 x\ 0$ = $2.5v^2$ J

But from the principle of conservation of energy mechanical energy

Mechanism energy at A = mechanical energy a t C

$258.7 = 2.5v^2$

v = $10.2ms^{-1}$.

(ii) Mechanical advantage at B, = m$\frac{v^2}{r} + mgh$ = 5 x $\frac{v^2}{2}$ + 5 x 9.81 x 2 = 258.7

v = 8ms

tension at B = m$\frac{v^2}{r}$ = $5\ x\ \frac{8^2}{2}$ =160N

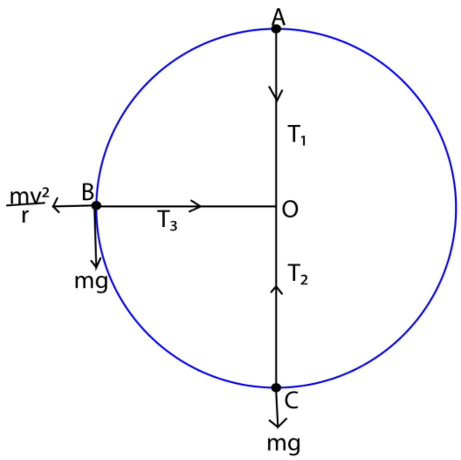
(iii) From a = $\frac{v^2}{r}$ = $\frac{8^2}{2}$ =$32ms^{-1}$

### Example 6

A particle of mass m describes a complete vertical inextensible string of length, r, given that the speed at the lowest point is twice the speed at highest point. Show

(i) The speed of the particle at the lowest point is = v= $4\sqrt{\frac{gr}{3}}$.

(ii) The tension in the string when the particle is at the highest point, T = $\frac{mg}{3}$




(i) Mechanical energy at A = mechanical energy at B = m$\frac{v^2}{r}$ + mg

Let the speed at A = u

The speed at C = 2u

⇨ $m\frac{u^2}{2} + mg.2r = m\frac{(2u)^2}{2} + mg\ x\ 0$

$u^2 + 4gr = 4vu^2$

$3u^2 = 4gr$

$u = 2\sqrt{\frac{gr}{3}}$.

At the lowest point velocity = 2u = $4\sqrt{\frac{gr}{3}}$.

(ii) Tension at the highest point = $m\frac{v^2}{r} - mg = \frac{4mgr}{3r} - mg = \frac{4mg-3mg}{3} = \frac{mg}{3}$

Exercise

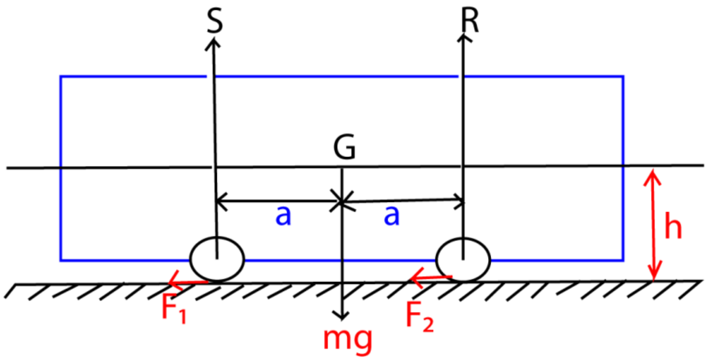
1. An object of mass 0.kg on the end of a string is whirled around in a horizontal circle of radius 2m, with a constant speed of $10ms^{-1}$. Find its angular velocity and the tension in the string [Ans. ω = $5rads^{-1}$, T = 25.5N)
2. A small ball of mass 0.1kg is suspended by an inextensible string of length 0.5m and is caused to rotate in a horizontal circle of radius 0.4m. find
   (i) The tension in the string [Ans. 1.3N]
   (ii) The period of rotation [ans. 1.2s]
3. A pendulum bob of mass 0.2kg is attached to one end of an inelastic string of length 1.2m. the bob moves in a horizontal circle with the string inclined at $30^0$ to the vertical. Calculate
   (i) Tension in the string [Ans. 2.27N]
   (ii) The period of motion [Ans. 2.02s]
4. The period of oscillation of a conical pendulum is 2.0s. if the string makes an angle $60^0$ to the vertical at the point of suspension, calculate the
   (i) vertical height of the point of suspension above the circle [ h = 0.994m]
   (ii) length of the string [ L= 1.99m]
   (iii) velocity of mass attached to the string. [v = $5.41ms^{-1}$]

## Application of circular motion

(i) motion of the car around a circular level track (unbanked track)

### (a) Overturning and toppling

Consider a car negotiating a bond on a level track. For a circular motion, the friction F1 and F2 provide the centripetal force.




Let 2a = the distance between the tyres,

h = height of the center of gravity of the car above the ground.

mg = weight of the car.

S and R = normal reaction on the tyres from the ground

Then,

$F_1 + F_2 = m\frac{mv^2}{r}$ ……………………………………………..(i)

Since the car does not move off the road, the sum upward force is equal to the sum of downward forces

R + S = mg …………………………………………………. (ii)

Taking moments about G

$Sa + F_1h + F2h = Ra$

$h(F_1 + F_2) = a(R - S)$

Substituting F1+ F2 from Eqn (i)

$m\frac{mv^2}{r}h = a(R\text{-}S)$

$(R\text{-} S) = m\frac{mv^2}{r}\ x\ \frac{h}{a}$………………………………………….. (iii)

Eqn. (ii) + Eqn. (iii)

$2R = mg + m\frac{mv^2}{r}\ x\ \frac{h}{a}$

$R = \frac{m}{2}\left(g + \frac{v^2}{ra}h\right)$

Eqn. (ii) – Eqn. (iii)

$2S = mg\ \text{-}\ m\frac{mv^2}{r}\ x\ \frac{h}{a}$

$S = \frac{m}{2}\left(g - \frac{v^2}{ra}h\right)$

If S = 0 , then the car is just about to topple/overturn

$0 = \frac{m}{2}\left(g - \frac{v^2}{ra}h\right)$

Either $\frac{m}{2} = 0$

or $\left(g - \frac{v^2}{ra}h\right) = 0$

$g = \frac{v^2}{ra}h$

But v = $\sqrt{\frac{gra}{h}}$

A car topples when v > $\sqrt{\frac{gra}{h}}$

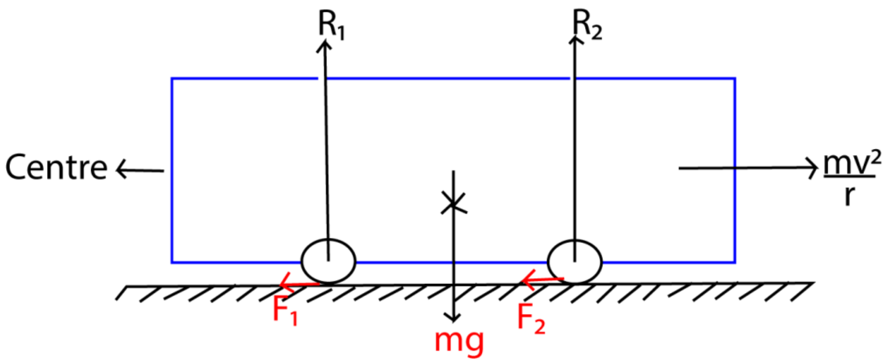
Condition for overturning/toppling

(i) The center of gravity of the car is high
(ii) If the bend is sharp
(iii) When the distance between the tyres is small

### (b) Sliding/ slipping/skidding

A vehicle skid when the available centripetal force is not enough to balance the centrifugal force. The vehicle fails to negotiate the curve and goes off the track outwards.

Consider a vehicle of mass, m, taking unbanked curve of radius, r, at speed, v.




$F_1 + F_2 = m\frac{v^2}{r}$

But $F_1 = \mu R_1$ and $F_2 = \mu R_2$

$\mu R_1 + \mu R_2 = m\frac{v^2}{r}$

$\mu (R_1 + R_2) = m\frac{v^2}{r}$

$\mu mg = m\frac{v^2}{r}$

$\mu g = \frac{v^2}{r}$

$v^2 = \mu gr$

v = $\sqrt{\mu gr}$ ( this the maximum velocity for which the vehicle does not skid/slip

A vehicle skid/ slips when v >$\sqrt{\mu gr}$

Conditions that lead to skidding

(i) Sharp bends
(ii) Slippery roads
(iii) Very high speed

**Example 6**

A car goes around unbanked curve at 15ms$^{-1}$, the radius of the curve is 60m. Find the least coefficient of kinetic friction that will allow the car to negotiate the curve without skidding.

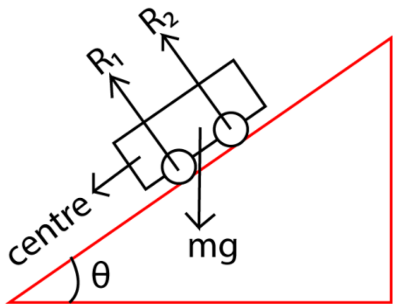
### Solution

$\mu \geq \frac{v^2}{rg} = \frac{15^2}{(60 \, x \, 9.81)} = 0.38$

## Motion of a car round a banked inclined track

### Angle of banking

Consider car negotiating a bend inclined at an angle θ to the horizontal. It is assumed that there is no tendency to slip at the wheels, therefore no frictional forces.




Resolving horizontally

$R_1\sin\theta + R_2\sin\theta = m\frac{v^2}{r}$ ................................(i)

Resolving vertically

$R_1\cos\theta + R_2\cos\theta = mg$ ................................(ii)

Eqn (i) ÷ Eqn (ii)

$\frac{R_1 \sin\theta + R_2 sin\theta}{R_1 \cos\theta + R_2 cos\theta} = {m\frac{v^2}{r}}/{mg}$

$\tan\theta = \frac{v^2}{rg}$

$v^2 = rgtan\,\theta$

$v = \sqrt{rgtan\,\theta}$

### Example 7

(a) A bend on a level road form a circular arc of radius 54m. Find the highest speed at which the car can travel around the bend without slipping occurring. If the coefficient of friction between the car tyres and the road surface is 0.3

(b) A car of mass 400kg turns a corner at 40kmhr$^{-1}$ without skidding but at 50kmh$^{-1}$, it skids off. If the corner forms an arc of radius 20m. Find the values between which the coefficient of friction between the wheels and the road surface lies.

Solution

(a) R = 54m, μ = 0.3

From v = $\sqrt{\mu gr}$

v = $\sqrt{(0.3 \, x \, 9.81 \, x \, 54}$ 12.60ms$^{-1}$

(b) Case (i)

Vmax = 40kmhr$^{-1}$ = $\frac{40 x 1000}{3600} = \frac{100}{9} ms^{-1}$

r = 20m, m = 400kg

From v = $\sqrt{\mu g r}$

$$\mu = \frac{v^2}{rg} = \frac{\left[\frac{100}{9}\right]^2}{20 x 9.81} = 0.629$$

Case (ii)

Vmax = 50kmhr$^{-1}$ = $\frac{50 x 1000}{3600} = \frac{125}{9} ms^{-1}$

From v = $\sqrt{\mu g r}$

$$\mu = \frac{v^2}{rg} = \frac{\left[\frac{125}{9}\right]^2}{20 x 9.81} = 0.983$$

Therefore, $0.629 \leq \mu \leq 0.983$

### Example 8

A car of mass 300kg travels at 100kmhr$^{-1}$ round unbanked curve of radius 200m.

(i) What is the minimum coefficient of sliding friction between the road and the tyres that will permit the car to negotiate the curve without skidding?

(ii) At what angle would the road be banked if there were no friction between the tyres and the road surface?

Solution

(i) From $\mu = \frac{v^2}{rg}$

v = 100kmhr$^{-1}$ 10= $\frac{250}{9} ms^{-1}$

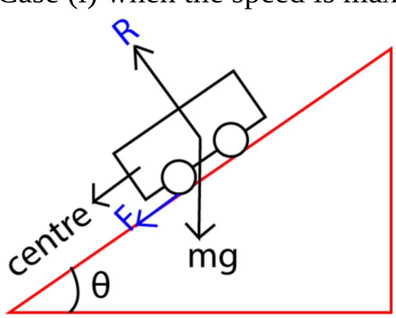
$$\mu = \frac{\left[\frac{250}{9}\right]^2}{200 x 9.81} = 0.393$$

From $\theta = \tan^{-1}\left[\frac{v^2}{rg}\right]$

$$\theta = tan^{-1}\left[\frac{\left(\frac{250}{9}\right)^2}{200 x 9.81}\right] = 21.5^0$$

(ii) Note that , all roads are banked and there is friction between the tyres and the road surface. The necessary centripetal force is provided by the horizontal component of normal reaction and friction force. There are two cases

Case (i) when the speed is maximum



Resolving horizontally

Rsinθ + Fcosθ = $m\frac{v^2}{r}$

But F = μR

Rsinθ + μRcosθ = $m\frac{v^2}{r}$ ……………………………(i)

Resolving vertically

Rcosθ -Fsinθ = $mg$

Rsinθ – μRsinθ = mg ……………………………(ii)

Eqn (i) ÷ Eqn (ii)

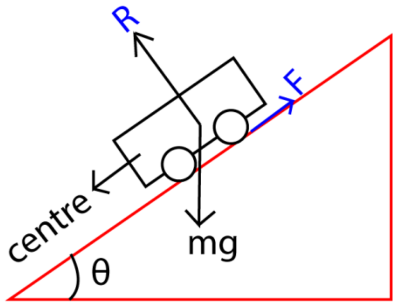
$$\frac{R\sin\theta + \mu R\cos\theta}{R\sin\theta - \mu R\sin\theta} = m\frac{v^2}{r}\Big/mg = \frac{v^2}{rg}$$

Dividing through by Rcosθ

$$\frac{\tan\theta + \mu}{1-\mu\tan\theta} = \frac{v^2}{rg}$$

$$v = \sqrt{\frac{rg\,(\tan\theta + \mu)}{1-\mu\tan\theta)}}$$

Case (ii) When the speed is minimum




Resolving horizontally

Rsinθ - Fcosθ = $m\frac{v^2}{r}$

But F = μR

Rsinθ - μRcosθ = $m\frac{v^2}{r}$ ……………………………(i)

Resolving vertically

Rcosθ +Fsinθ = $mg$

Rsinθ + μRsinθ = mg ……………………………(ii)

Eqn (i) ÷ Eqn (ii)

$$\frac{R\sin\theta - \mu R\cos\theta}{R\sin\theta + \mu R\sin\theta} = m\frac{v^2}{r}\Big/mg = \frac{v^2}{rg}$$

Dividing through by Rcosθ

$$\frac{\tan\theta - \mu}{1+\mu\tan\theta} = \frac{v^2}{rg}$$

$$v = \sqrt{\frac{rg\,(\tan\theta - \mu)}{1+\mu\tan\theta)}}$$