ALGEBRA · 2

COMPLEX NUMBERS

A complex number is a number containing both imaginary number ~~real and imaginary~~ ~~parts~~. An imaginary number is the square root of a neg No.

A complex number can be represented as $z = x + yi$ where x and y are real numbers. x is the real part of z while y is the imaginary part and $i = \sqrt{-1}$

i iota for the root of -1

<u>NB</u>:

from $i = \sqrt{-1}$

integrals powers of iota

$$i^2 = -1$$

$$\Rightarrow \quad i^3 = i^2 i = -i$$

$$i^4 = i^2 \cdot i^2 = 1$$

$$i^5 = i^4 \cdot i = i$$

$$i^6 = i^5 \cdot i = -1$$

$$\frac{1}{i^3} = \frac{i}{i^4} = i$$

A number $z = a + ib$ wher a and b are real is a comblex No

The idea of complex numbers can be used to solve quadratic equations with non real roots ie equations of the form $ax^2 + bx + c = 0$, where $b^2 < 4ac$.

If the imaginary part is zero the comp is known as a purely real No.

~~facts~~ about complex

Algebra of complex NRs ~~Numbers~~

* A complex number Z has two parts ie the real part and the imaginary part which are represented as $Re(z)$ and $Im(z)$ respectively.

* $\bar{z} = a - bi$ is the complex conjugate of the complex number $z = a + bi$ and

$$z\bar{z} = [Re(z)]^2 + [Im(z)]^2$$

<u>proof</u>

let $z = x + yi$ and $\bar{z} = x - yi$

$$\Rightarrow z\bar{z} = (x + yi)(x - yi)$$

$$= x^2 - (yi)^2$$

$$= x^2 + y^2$$

$$\therefore z\bar{z} = [Re(z)]^2 + [Im(z)]^2$$

* When a complex number appears in the denominator, it is always removed by rationalising the denominator. To rationalise the denominator, we multiply both the numerator and denominator by the conjugate of the denominator.

ie $$\frac{x + yi}{a + bi} = \frac{(x + yi)(a - bi)}{(a + bi)(a - bi)} = \frac{(x + yi)(a - bi)}{a^2 + b^2}$$

* complex numbers are added or subtracted by adding or subtracting the corresponding parts.

ie $(a + bi) \pm (x + yi) = (a \pm b) + (b \pm y)i$

* The reciprocal of a complex number $z = x + yi$ is given by

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$$

$$= \frac{x - yi}{x^2 + y^2}$$

Examples

① Given that $z_1 = 2+3i$ and $z_2 = 4+6i$ find;

(i) $z_1 + z_2$

(ii) $z_1 - z_2$

(iii) $z_1 z_2$

② Work out

(i) $(3+2i) - (3-2i)$

(ii) $(2-7i)(3+2i)$

(iii) $(2-3i)(6+2i)$

(iv) $\frac{5+2i}{1-3i}$

(v) $\frac{3+4i}{1+2i}$

③ Given that $z_1 = 2+2i$ and $z_2 = 1+i$ express $\frac{z_1 z_2^2}{z_1 + z_2}$ in the form $a+bi$

NB:

If a complex number $z = x+yi$ is a root of an equation, then its conjugate $\bar{z} = x-yi$ is also a root of that equation.

· Divide $z^4 + 3z^2 - 6z + 10$ by $z = 1+i$

$(1+i)(1+i) = 1+2i-1$

	z^4	z^3	z^2	z^1	z^0
$1+i$	1	0	3	-6	10
		$1+i$	$2i$	$1+5i$	-10
+	1	$1+i$	$3+2i$	$-5+5i$	0

Examples:

① Solve the equations

(i) $z^2 + 2z + 6 = 0$

(ii) $z^2 - 4z + 5 = 0$

Soln:

$$z = \frac{4 \pm \sqrt{16-20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm \sqrt{4}i}{2}$$

$\therefore z = 2+i$ or $z = 2-i$

② Form a quadratic equation whose root is;

(i) $4+i$

(ii) $2-3i$

Soln

If $4+i$ is a root, then $4-i$ is also a root

Sum of roots $= (4+i)+(4-i) = 8$

Product of roots $= 4^2 + 1^2 = 17$

$\therefore z^2 - 8z + 17 = 0$ is the required equation.

③ If one root of the equation $z^3 - 7z^2 + 19z - 13 = 0$ is real, find the other roots.

④ Show that $1+i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$ hence find the other roots.

Exercise

① If $2+i$ is a root of the equation $2z^3 - 9z^2 + 14z - 5 = 0$, find the other roots.

② Show that $2+3i$ is a root of the equation

$z^4 - 5z^3 + 18z^2 - 17z + 13 = 0$ hence find the other roots.

③ Show that $z + 4i$ is a root of the equation $z^4 - 4z^3 + 21z^2 - 4z + 20 = 0$ hence find the other roots

Equality of complex numbers.

Complex numbers are equal only if their corresponding parts are equal i.e the real part of one complex number is equal to the real part of the other and so is the imaginary part.

eg if $z_1 = a + bi$ and $z_2 = x + yi$ are equal then $a = x$ and $b = y$.

Examples

① Find a and b given that $i(a+2b) + 2a+b = 7+8i$

($a = 2$ and $b = 3$)

② Given that $a + bi = \frac{1}{x+yi}$ Show that $(a^2+b^2)(x^2+y^2) = 1$

③ Solve the equation $(z + 2\bar{z})z = 5 + 2z$ given that $\bar{z}$ is the complex conjugate of the complex number z.

$$\begin{bmatrix} z = 1+2i & \text{or } z = 1-2i \\ \bar{z} = 1-2i & \text{or } \bar{z} = 1+2i \end{bmatrix}$$

Exercise

① Find the square root of $(12i - 5)$

② Given that, $(x+y) + i(x-y) = 3+i$, find x and y.

($x = 2$ and $y = 1$)

③ Given that x and y are real, solve the equation;

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

$$\frac{5}{x+iy} + \frac{2}{1+3i} = 1$$

④ Find a and b if; $z^4 - 2z^2 + 5 = (z^2 - a)^2 + b$

⑤ If $\bar{z}$ is the complex conjugate of a complex number z. Solve the following equations.

(i) $\frac{z}{z+1} = 1 + 2i$

(ii) $\frac{z+i}{z-3} = \frac{1+i}{z+1}$

(iii) $\frac{1}{z} + \frac{2}{\bar{z}} = 1 + i$

⑥ Given that the complex number z and its conjugate satisfy the equations, Find z.

(a) $z\bar{z} + 2iz = 12 + 6i$

(b) $z\bar{z} + 3z = 34 - 12i$

(c) $z\bar{z} - 5iz = 5(9 - 7i)$

Find the values of x and y in $\frac{x}{3i+2} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$

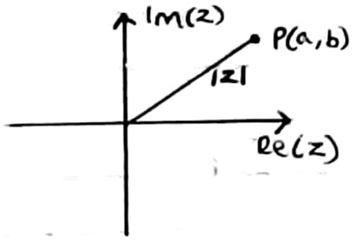
Find z if $\left(\frac{z+1}{z-1}\right) = -1$

Graphical representation of complex numbers

Complex numbers are represented graphically on argand diagrams.

An argand diagram is a cartesian plane whose x- and y- axes have been replaced by the Re- and Im- axes of the complex plane.

If $z = a + bi$ and a point $P(a, b)$ is the argand diagram argand diagram then OP, where O is the origin represents the complex number z.

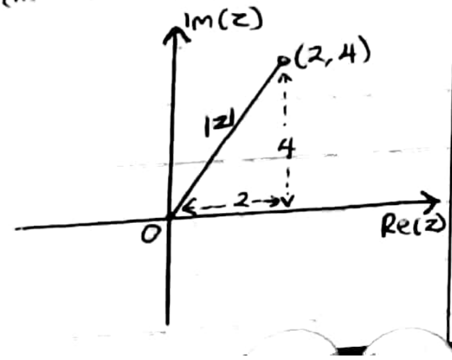


Examples

Represent each of the following complex numbers on an argand diagram.

(I) $z = 2 + 4i$

(II) $z = -3 + 2i$



Modulus of a complex number

The modulus of a complex number $z = a + bi$ is denoted as $|z|$ and defined as $|z| = \sqrt{a^2 + b^2}$

NB:

$$|z_1 z_2| = |z_1| + |z_2|$$

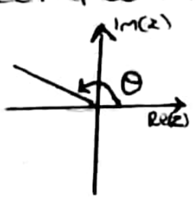
$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

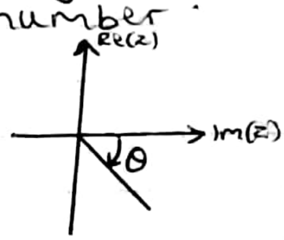
$$|z^n| = |z|^n$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Argument (principal number) of a complex number

This is the angle between the positive real axis and the line representing the complex number.





The argument of a complex number $z = x + yi$ is denoted as $\arg(z)$ and defined as $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

NB:

(I) $-\pi \leq \arg z \leq \pi$

(II) $\arg\left(z_1/z_2\right) = \arg(z_1) - \arg(z_2)$

(III) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(IV) $\arg\left(\frac{1}{z}\right) = -\arg(z)$

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Examples

① Find the modulus and argument of each of the complex numbers below.

(i) $Z = 1 + i$

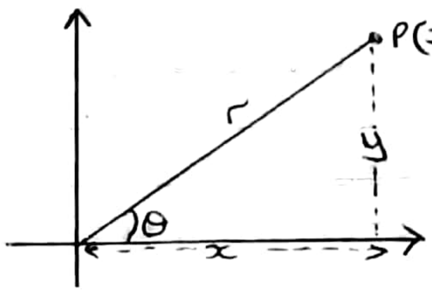
(ii) $Z = -2 + 3i$

② Given that $Z_1 = 1 - i$ and $Z_2 = 2 - i$, find the modulus and argument of Z_1/Z_2

③ Given that $Z_1 = 2 + i$ and $Z_2 = 1 - i$, find the modulus and argument of Z_1^2/Z_2

Polar notation of a complex number

consider the argand diagram below showing a complex number $Z = x + yi$ whose modulus and argument are r and θ respectively.



$\cos\theta = \frac{x}{r}$; $x = r\cos\theta$

$\sin\theta = \frac{y}{r}$; $y = r\sin\theta$

$\Rightarrow x + yi = r\cos\theta + ir\sin\theta$

$\therefore Z = r(\cos\theta + i\sin\theta)$

Examples

① Express each of the following complex numbers in polar form.

(i) $Z = 1 + i$

(ii) $Z = -3i$

Soln

(i) $|Z| = \sqrt{2}$ units

$\arg Z = \frac{1}{4}\pi$

$\therefore Z = \sqrt{2}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

(ii) $Z = 3(\cos\frac{1}{2}\pi - i\sin\frac{1}{2}\pi)$

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② If $Z_1 = 2 + i$ and $Z_2 = 1 - i$, express Z_1^2/Z in polar form.

③ Express $i/4+6i$ in modulus argument form.

④ Express $Z_1 = 4i$ and $Z_2 = 2 - 2i$ in polar form hence or otherwise find Z_1/Z_1Z_2

⑤ Given that $Z = \frac{(2-i)(5+12i)}{(1+2i)^2}$, Find the modulus and argument of Z hence express Z in polar form.

⑥ Express the complex number $Z = \frac{(3i+1)(i-2)}{i-3}$ in polar form.

Demoivre's theorem

According to De moivre;

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

proof

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos 2\theta + i\sin 2\theta$

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta$

$= \cos 3\theta + i\sin 3\theta$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

hence it holds for all positive integral values of n

Now consider $n = -m$

$(\cos\theta + i\sin\theta)^{-m} = \dfrac{1}{(\cos m\theta + i\sin m\theta)}$

$= \dfrac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta + \sin^2 m\theta)}$

$= \cos(-m\theta) + i\sin(-m\theta)$

$= (\cos\theta + i\sin\theta)^{-m}$

$\therefore$ It holds for all integral values of n.

Examples.

① Use De Moivre's theorem $\cos\pi + i\sin\pi$ in the form $a + bi$

② Express $z_1 = 3 + i$ and $z_2 = 1 + i$ in polar form hence find $(z_1)^2/(z_2)^3$ in the form $a + bi$

③ Use De Moivre's theorem to simplify.

(i) $\dfrac{(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)}{(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta)}$

(ii) $\dfrac{\left(\cos\pi/17 + i\sin\pi/17\right)^{8}}{\left(\cos\pi/17 - i\sin\pi/17\right)^{9}}$ (iii) $\dfrac{\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^{8}}{\left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)^{3}}$

4. Use Demoivre`s theorem to prove that:

(i) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(iv) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ hence solve the equation

(v) $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^2\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

5. Prove that (i) $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

(ii) $\cos^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

6. Given that $z = \cos\theta + i\sin\theta$ show that $\left(z + \frac{1}{z}\right)^3 = 8\cos^3\theta$. By expanding $\left(z + \frac{1}{z}\right)^3$, prove that $2\cos 3\theta + 6\cos\theta = 8\cos^3\theta$

7. Use De Moivre`s theorem to show that $2\cos\theta = z + z^{-1}$ and $2\cos n\theta = z^n + z^{-n}$ where n is a positive integer hence solve the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$

8. Use De Moivre`s theorem to show that: $\dfrac{\cos 5x}{\cos x} = 1 - 12\sin^2 x + 16\sin^4 x$.

9. Use De Moivre`s theorem to show that $\tan 4\theta = 4\tan\theta - 4\tan^3\theta$ hence solve $\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$

Complex roots of unity

If $Z^n = r(\cos\theta + i\sin\theta)$ then $Z = (r(\cos\theta + i\sin\theta))^{\frac{1}{n}}$

$= r^{1/n}(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta)$

but $\sin\theta = \sin(2\pi k + \theta)$ and $\cos\theta = \cos(2\pi k + \theta)$

where $k = \{0, 1, 2, \ldots\}$.

$\therefore Z_k = r^{1/n}\left(\cos\left(\frac{2\pi k + \theta}{n}\right) + i\sin\left(\frac{2\pi k + \theta}{n}\right)\right)$ where

$k = \{0, 1, 2, 3, \ldots, n-1\}$

Examples

① Find the square root of the following complex numbers in the form $a + bi$.

(i) $4i$

(ii) $12i - 5$.

② Find the forth root of $4 + 3i$

③ Find the complex cube root of 1.

$z^3 - 1 = 0$

$\left[1,\ -\frac{1}{2} + \frac{1}{2}i\sqrt{3},\ -\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right]$

NB.

If ω is used to denote the cube root of unity then $\omega^2 = \frac{1}{2}(-1 - i\sqrt{3})$ and

$\omega^4 = \omega^3 \cdot \omega = \omega$

$\omega^5 = \omega^3 \cdot \omega^2 = \omega^2$

$1 + \omega + \omega^2 = 0$

Examples

① If ω is a complex cube root of unity form a quadratic equation whose roots are ω and $\frac{1}{\omega}$

Soln.

Sum of roots $= \omega + \frac{1}{\omega}$

$= \frac{\omega^2 + 1}{\omega} = \frac{-\omega}{\omega} = -1$ (since $\omega^2 + 1 = -\omega$)

Product of roots $= 1$

$\therefore Z^2 + Z + 1 = 0$ is the required equation.