ALGEBRA

EQUATIONS

(a) QUADRATIC EQUATIONS

These are equations of the form $ax^2+bx+c=0$ where a, b and c are constants and $a\neq 0$.

The expression ax^2+bx+c where a, b and c are constants and $a\neq 0$ is called a quadratic expression.

Minimum and maximum values of a quadratic expression.

(i) Maximum value

This is the greatest value of a quadratic expression.

A quadratic expression will be a maximum only if a is negative.

Examples:

① Find the maximum value of $5-2x-4x^2$ and state the value of x for which it occurs.

Soln

$$5-2x-4x^2 = -4x^2-2x+5$$

$$= -4\left[x^2+\tfrac{1}{2}x-\tfrac{5}{4}\right]$$

$$= -4\left[x^2+\tfrac{1}{2}x-\left(\tfrac{1}{4}\right)^2-\tfrac{5}{4}-\left(\tfrac{1}{4}\right)^2\right]$$

$$= -4\left[\left(x+\tfrac{1}{4}\right)^2-\tfrac{21}{16}\right]$$

$$= -4\left(x+\tfrac{1}{4}\right)^2+\tfrac{21}{4}$$

$\therefore$ The maximum value of $5-2x-4x^2$ is $21/4$ and it occurs when $x=-1/4$.

(ii) Minimum value

This is the least value of a quadratic expression.

A quadratic expression will be a minimum only if a is positive.

Examples

① Find the minimum value of the expression x^2+3x+4 and state the value of x for which it occurs.

soln

$$x^2+3x+4 = x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+4$$

$$= \left(x+\frac{3}{2}\right)^2+\frac{7}{4}$$

∴ The minimum value of x^2+3x+4 is $7/4$ and it occurs when $x = -3/2$.

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② Show that $3x^2+10x+9$ can not be negative hence deduce its least value.

soln

$$3x^2+10x+9 \equiv 3\left(x^2+\frac{10}{3}x+3\right)$$

$$= 3\left(x^2+\frac{10}{3}x+\left(\frac{10}{6}\right)^2-\left(\frac{10}{6}\right)^2+3\right)$$

$$= 3\left(\left(x+\frac{5}{3}\right)^2+\frac{2}{9}\right)$$

$$= 3\left(x+\frac{5}{3}\right)^2+\frac{2}{3}$$

Since $\left(x+\frac{5}{3}\right)^2$ can not be negative, the expression $3x^2+10x+9$ can never be negative.

The least value of $\left(x+\frac{5}{3}\right)^2$ is zero hence the least value of $3x^2+10x+9$ is $2/3$.

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③ Show that the expression $a^2+b^2+c^2-ac-ab-bc$ can never be negative hence deduce the circumstance under which the value of the expression is zero.

Soln

$$a^2+b^2+c^2-ac-ab-bc = \frac{1}{2}\left[2a^2+2b^2+2c^2-2ac-2ab-2bc\right]$$
$$= \frac{1}{2}\left[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(a^2+c^2-2ac)\right]$$
$$= \frac{1}{2}\left[(a-b)^2+(a-c)^2+(b-c)^2\right]$$

Since all the terms involved are squares, the expression can never be a negative.
The expression is zero when $a=b=c$.

Exercise

(1) Find the minimum or maximum value of each of the following expressions and in each case state the value of x for which it occurs.

(I) x^2+4x-8 (II) $4-2x^2+3x$

(III) $4+6x-3x^2$ (IV) $2x^2+7x-16$

(V) $12x-2x^2+7$ (VI) $7-3x^2-2x$

(VII) $2x^2+7x+16$ (VIII) $4x^2-5x+2$

SOLVING QUADRATIC EQUATIONS

Quadratic equations can be solved by applying either the factorisation method, method of completing squares, the quadratic formula or graphical method.

(a) Factorisation method.

This method is applied to quadratic equations whose quadratic expressions can be factorised completely.

Examples.

(1) Solve the following quadratic equations.

(I) $x^2 + 2x - 8 = 0$ (II) $3x^2 + 7x - 6 = 0$

Soln

(I) $x^2 + 2x - 8 = 0$ $\qquad -8 \to 4, -2$

$$x^2 + 4x - 2x - 8 = 0$$
$$x(x+4) - 2(x+4) = 0$$
$$(x-2)(x+4) = 0$$

$\therefore$ either $x - 2 = 0$; $x = 2$

or $x + 4 = 0$; $x = -4$

(II) $3x^2 + 7x - 6 = 0$ $\qquad -18 \to 9, -2$

$$3x^2 + 9x - 2x - 6 = 0$$
$$3x(x+3) - 2(x+3) = 0$$
$$(3x-2)(x+3) = 0$$

$\therefore$ either $x + 3 = 0$; $x = -3$

or $3x - 2 = 0$; $x = 2/3$

(b) Completing Squares

This method can be used to solve any quadratic equation.

Examples

(I) Solve the equation $x^2 + 5x + 6 = 0$

Soln

$$x^2 + 5x + 6 = 0$$
$$x^2 + 5x = -6$$
$$x^2 + 5x + \left(\tfrac{5}{2}\right)^2 = -6 + \left(\tfrac{5}{2}\right)^2$$
$$\left(x + \tfrac{5}{2}\right)^2 = \tfrac{1}{4}$$
$$x + 5/2 = \pm \tfrac{1}{2}$$
$$x = \pm \tfrac{1}{2} - \tfrac{5}{2}$$

$\therefore$ $x = -2$ or $x = -3$

modification of the method of completing squares.

Using this method, any quadratic equation can be solved by applying the equation;

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The expression b^2-4ac is called the discriminant of the quadratic equation. ie

- if $b^2-4ac > 0$, the quadratic equation will have two distinct real roots.
- If $b^2-4ac = 0$, the quadratic equation is a perfect square and so has two equal roots.
- If $b^2-4ac < 0$, the quadratic equation will have complex roots.

<u>Derivation of the quadratic formular</u>

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -c/a$$

$$x^2 + \frac{b}{a}x^2 + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - c/a$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2-4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

<u>Examples.</u>

① Solve the equation $x^2 - x - 6 = 0$

<u>Soln</u>

$a = 1$, $b = -1$ and $c = -6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 + (4 \times 6)}}{2}$$

$$= \frac{1 \pm \sqrt{25}}{2}$$

$$= \frac{1 \pm 5}{2}$$

$\therefore\ x = 3$ or $x = -2$

Exercise

① Solve the following quadratic equations:

(a) $2x^2 - 3x - 7 = 0$ (b) $2x^2 + 3x - 2 = 0$

(c) $3x^2 + 10x = x + 24$ (d) $\frac{1}{3x-4} + \frac{x}{x+1} = 1$

ROOTS OF A QUADRATIC EQUATION.

Suppose α and β are the roots of the equation $ax^2 + bx + c = 0$, then $x = \alpha$ and $x = \beta$

$\Rightarrow x - \alpha = 0$ and $x - \beta = 0$.

It follows that $(x - \alpha)(x - \beta) = 0$

$(x - \alpha)$ and $(x - \beta)$ are the factors of the equation $ax^2 + bx + c = 0$.

NB:

On expanding $(x - \alpha)(x - \beta)$ we obtain;

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$(\alpha + \beta)$ is the sum of the roots.

$\alpha\beta$ is the product of the roots.

$\Rightarrow$ The quadratic equation is in the form $x^2 -$ (sum of roots)x + product of roots $= 0$

$\therefore\ x^2 - (\alpha + \beta)x + \alpha\beta = 0$ is the general equation of a quadratic equation whose roots are α and β.

Standardising a quadratic equation

The quadratic equation $ax^2+bx+c=0$ is said to be in standard form if a, the coefficient of x^2, is 1.

$\therefore$ The standard form of $ax^2+bx+c=0$ is $x^2+\frac{b}{a}x+\frac{c}{a}=0$.

$\Rightarrow \alpha+\beta = -b/a$ and $\alpha\beta = c/a$.

Examples

① State the product and sum of the roots of the equation $3x^2+9x+7=0$

Soln

$3x^2+9x+7=0$

$\Rightarrow x^2+3x+\frac{7}{3}=0$

$\therefore$ Sum of roots $= -3$

Product of roots $= 7/3$

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② Deduce the equation whose roots are 3 and -2.

Soln

Sum of roots $= 3+(-2) = 1$

Product of roots $= 3\times(-2) = -6$

$\therefore x^2-x-6=0$ is the required equation.

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③ If α and β are the roots of the equation $3x^2+4x-5=0$, evaluate;

(a) $\frac{1}{\alpha}+\frac{1}{\beta}$ (b) $\alpha^2+\beta^2$ (c) $\alpha^2-\beta^2$

(d) $\alpha^3+\beta^3$ (e) $\alpha^3-\beta^3$

Soln

(a) $3x^2+4x-5=0$

$\Rightarrow x^2+\frac{4}{3}x-\frac{5}{3}=0$

$(\alpha+\beta) = -4/3$ and $\alpha\beta = -5/3$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$
$$= \left(\frac{-4}{3}\right) / \left(-\frac{5}{3}\right)$$
$$= \frac{4}{5}$$

(b) $$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$= \left(-\frac{4}{3}\right)^2 - 2\left(-\frac{5}{3}\right)$$
$$= \frac{16}{9} + \frac{10}{3}$$
$$= \frac{46}{9}$$

(c) $$\alpha^2-\beta^2 = (\alpha+\beta)(\alpha-\beta)$$
$$= (\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$
$$= -\frac{4}{3}\sqrt{\left(-\frac{4}{3}\right)^2-4\left(-\frac{5}{3}\right)}$$
$$= -\frac{8\sqrt{19}}{9}$$
$$= -3.8746$$

(d) $$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$
$$= \left(-\frac{4}{3}\right)^3 - 3\left(-\frac{5}{3}\right)\left(-\frac{4}{3}\right)$$
$$= -\frac{64}{27} - \frac{60}{9}$$
$$= -\frac{244}{27}$$
$$= -9.037$$

(e) $$\alpha^3-\beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$$
$$= (\alpha-\beta)\left[(\alpha+\beta)^2-2\alpha\beta+\alpha\beta\right]$$
$$= (\alpha-\beta)\left[(\alpha+\beta)^2-\alpha\beta\right]$$
$$= \left(\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right)\left[(\alpha+\beta)^2-\alpha\beta\right]$$
$$= \left(\sqrt{\left(-\frac{4}{3}\right)^2-4\left(-\frac{5}{3}\right)}\right)\left[\left(-\frac{4}{3}\right)^2-\frac{5}{3}\right]$$
$$= 0.32288$$

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④ The roots of the equation $2x^2-7x+4=0$ are α and β. Find the equation whose roots are α/β and β/α.

Soln

$$x^2 - \frac{7}{2}x + 2 = 0$$

$$\Rightarrow \alpha+\beta = \frac{7}{2} \quad \text{and} \quad \alpha\beta = 2$$

let the roots of the new equation be α' and β'

$$\Rightarrow \alpha' + \beta' = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$= \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= 33/8$$

$$\alpha'\beta' = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)$$

$$= 1$$

$\therefore$ $8x^2 - 33x + 8 = 0$ is the required equation.

⑤ The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha+2$. Find the possible values of α and p.

Soln

$$x^2 - px + 8 = 0$$

sum of roots $= p$

$$\Rightarrow 2\alpha + 2 = p \text{ ---- (I)}$$

Product of roots $= 8$

$$\Rightarrow \alpha(\alpha+2) = 8$$

$$\alpha^2 + 2\alpha = 8 \text{ ---- (II)}$$

From (II), $\alpha = -4$ or $\alpha = 2$

when $\alpha = 2$, $p = (2\times2)+2 = 6$

when $\alpha = -4$, $p = (2\times-4)+2 = -6$

$\therefore$ either $\alpha = 2$ and $p = 6$ or $\alpha = -4$ and $p = -6$

⑥ The roots of the equation $ax^2+bx+c=0$ differ by 4. Prove that $b^2-4ac = 16a^2$.

Soln

let the roots be α and β

$\Rightarrow \alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = c/a$

But $\alpha - \beta = 4$

$\Rightarrow (\alpha-\beta)^2 = 16$

$(\alpha+\beta)^2 - 4\alpha\beta = 16$

$\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) = 16$

$\frac{b^2}{a^2} - \frac{4c}{a} = 16$

$\therefore b^2 - 4ac = 16a^2$.

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7 The roots of the equation $x^2 + 2px + q = 0$ differ by 8. Show that $p^2 - 16 = q$.

<u>Soln</u>

let the roots be α and β

$\Rightarrow \alpha+\beta = 2p$ and $\alpha\beta = q$.

But $\alpha - \beta = 8$

$\Rightarrow (\alpha-\beta)^2 = 64$

$(\alpha+\beta)^2 - 4\alpha\beta = 64$

$4p^2 - 4q = 64$

$p^2 - q = 16$

$\therefore p^2 - 16 = q$

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8 Prove that if the sum of the reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is 1 then $b + c = 0$. If in addition, one root of the equation is twice the other, find one set of values of a, b and c.

<u>Soln</u>

let the roots be α and β

$\Rightarrow \alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = c/a$

But $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

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$$\frac{\alpha+\beta}{\alpha\beta} = 1$$

$$\frac{-b/a}{c/a} = 1$$

$$c = -b$$

$$\therefore b + c = 0$$

In addition, $\alpha = 2\beta$

$$\Rightarrow \frac{\beta + 2\beta}{2\beta^2} = 1$$

$$3\beta = 2\beta^2$$

$$\beta = 3/2 \text{ and } \alpha = 3$$

$$\alpha + \beta = \frac{3}{2} + 3 = 9/2$$

$$\alpha\beta = \frac{3}{2} \times 3 = 9/2$$

$\Rightarrow 2x^2 - 9x + 9 = 0$ is the quadratic equation.

$\therefore a = 2$, $b = -9$ and $c = 9$

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(9) If the equation $a^2x^2 + 6abx + ac + 8b^2 = 0$ has equal roots, prove that $ac(x+1)^2 = 4b^2x$ also has equal roots.

Soln

let the root of $a^2x^2 + 6acx + ac + 8b^2 = 0$ be α

$\Rightarrow$ sum of roots $= 2\alpha = -\frac{6b}{a}$ ---- (I)

product of roots $= \alpha^2 = \frac{ac + 8b^2}{a^2}$ --- (II)

From (I), $\alpha = -\frac{3b}{a}$

Substitute for α in (II)

$$\frac{9b^2}{a^2} = \frac{ac + 8b^2}{a^2}$$

$$b^2 = ac$$

Substitute for b^2 in $ac(x+1)^2 = 4b^2x$

$$\Rightarrow ac(x+1)^2 = 4acx$$

$$(x+1)^2 = 4x$$

$$x^2 + 2x + 1 = 4x$$

$$x^2-2x+1=0$$
$$\therefore (x-1)^2=0$$

Since $(x-1)^2$ is a perfect square, there is only one possible value of x hence $ac(x+1)^2=4b^2x$ also has equal roots.

(10) Given that α and β are the roots of the equation $x^2+5x+6=0$, form an equation whose roots are $\frac{1}{1-\alpha}$ and $\frac{1}{1-\beta}$

Soln

$$x^2+5x+6=0$$
$$\Rightarrow \alpha+\beta=-5 \quad \text{and} \quad \alpha\beta=6$$

Let α' and β' be the roots of the new equation.

$$\Rightarrow \alpha'+\beta' = \frac{1}{1-\alpha}+\frac{1}{1-\beta}$$
$$= \frac{(1-\beta)+(1-\alpha)}{(1-\alpha)(1-\beta)}$$
$$= \frac{2-(\alpha+\beta)}{1-(\alpha+\beta)+\alpha\beta}$$
$$= \frac{2-(-5)}{1-(-5)+6} = \frac{7}{12}$$

$$\alpha'\beta' = \left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-\beta}\right)$$
$$= \frac{1}{1-(\alpha+\beta)+\alpha\beta}$$
$$= \frac{1}{12}$$

$$\Rightarrow x^2-\frac{7}{12}x+\frac{1}{12}=0$$

$\therefore 12x^2-7x+1=0$ is the required equation.

⑪ The roots of the quadratic equation $x^2-2x+2=0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$. Show that one of the equations whose roots are α and β is $x^2+4=0$

Soln

$$x^2-2x+2=0$$

Product of roots, $\sqrt{\alpha}\sqrt{\beta} = 2$

$$\sqrt{\alpha\beta} = 2$$

$$\Rightarrow \alpha\beta = 4.$$

Sum of roots, $\sqrt{\alpha}+\sqrt{\beta} = 2$

$$(\sqrt{\alpha}+\sqrt{\beta})^2 = 4$$

$$\alpha+2\sqrt{\alpha\beta}+\beta = 4$$

$$\alpha+4+\beta = 4$$

$$\alpha+\beta = 0$$

$$\therefore x^2+4=0$$

Exercise

① Express the following in terms of $\alpha+\beta$ and $\alpha\beta$.

(a) $\alpha^4+\beta^4$ (b) $\alpha^4-\beta^4$

② Prove that if the difference between the roots of the equation $ax^2+bx+c=0$ is 1 then $a^2=b^2-4ac$.

③ One root of the equation $ax^2+bx+c=0$ is the square or the other. Show that;

(I) $b^3=ac(3b-c-a)$ (II) $c(a-b)^3=a(c-b)^3$

④ Prove that if the sum of the squares of the roots of the equation $ax^2+bx+c=0$ is 1 then $b^2=2ac+a^2$.

⑤ The roots of the equation $x^2-px+q=0$ are α and β. Form a quadratic equation whose roots are $(\alpha+1)$ and $(\beta+1)$

(6) The roots of the equation $x^2-12x+2=0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$. Show that one of the equations whose roots are α and β is $x^2-8x+4=0$.

(7) Given that the roots of the equation $x^2-2x+10=0$ are α and β, determine the equation whose roots are $\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$

(8) One root of the quadratic equation $ax^2+bx+c=0$ is three times the other. Show that $3b^2-16ac=0$.

(9) Prove that if the difference between the roots of the equation $ax^2+bx+c=0$ is 6, then $a^2=\frac{b^2-4ac}{36}$

(10) If the roots of the equation $x^2+px+q=0$ are α and β, find the equation whose roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

(11) Given that the roots of the equation $x^2+px+q=0$ are α and β, express $(\alpha-\beta^2)(\beta-\alpha^2)$ in terms of p and q hence deduce that for one root to be the square of the other, $p^3-3pq+q=0$ must hold.

(12) The roots of the quadratic equation $x^2+px+q=0$ are α and β. Show that the equation whose roots are $\alpha^2-q\alpha$ and $\beta^2-q\beta$ is $x^2-(p^2+pq-2q)x+q^2(q+p+1)=0$.

(13) Find the equation whose roots are

the squares of the roots of the equation $3x^2+5x-1=0$.

(14) If the roots of the equation $x^2-bx+c=0$ are $\sqrt{\alpha}$ and $\sqrt{\beta}$, show that;

(I) $\alpha+\beta=b^2-2c$

(II) $\alpha^2+\beta^2=(b^2-2c-\sqrt{2c})(b^2-2c+\sqrt{2c})$

(15) Find the values of m for which the equation $x^2+(m+3)x+4m=0$ has equal roots. For what values of m is the sum of the roots equal to zero?

(16) The roots of the equation $3x^2-ax+6b=0$ are α and β. Find the condition for one root to be;

(I) twice the other $(81b=a^2)$ (II) the cube of the other $(a^4-648(b-1)b^2-18(4a^2+9)b=0)$

(17) Given that the roots of the quadratic equation $\frac{1}{x}+\frac{1}{1+x}-\frac{1}{2}=0$ are α and β. Form a quadratic equation whose roots are $(\alpha-\beta)^2$ and $\alpha^3+\beta^3$

$$(x^2-62x+765=0)$$

(18) Given that α and β are the roots of the equation $x^2+px+q=0$,

(I) Show that $(\alpha^2+1)(\beta^2+1)=(q-1)^2+p^2$

(II) Find a quadratic equation whose roots are $\frac{\alpha}{\alpha^2+1}$ and $\frac{\beta}{\beta^2+1}$

$$[((q-1)^2+p^2)x^2+p(1+q)x+q=0]$$

(19) Given that the roots of the equation $x^2+px+q=0$ are α and β, deduce the condition for one root to be three times the other hence show that $p^2:q=16:3$

(20) Prove that the roots of the equation

$(\alpha+3)x^2+(6-2\alpha)x+\alpha-1=0$ are real if and only if α is not greater than $3/2$. Find the value of α if one root is six times the other.

(21) Show that if $x^2+bx+c=0$ and $x^2+px+q=0$ have a common root, then $(c-q)^2=(b-p)(cp-bq)$

(22) If the roots of the equation $x^2+bx+c=0$ are α and β and the roots of the equation $x^2+\lambda bx+\lambda^2c=0$ are γ and δ, show that the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$ is given by

$$x^2-\lambda b^2x+2\lambda^2c(b^2-2c)=0$$

(23) Given that α and β are roots of the equation $ax^2+bx+c=0$, determine the equation whose roots are $\alpha+\beta$ and $\alpha^3+\beta^3$ hence or otherwise solve $\alpha+\beta=2$ and $\alpha^3+\beta^3=26$ simultaneously.

(24) If α and β are the roots of the equation $x^2-px+q=0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ and $\frac{\beta^3-1}{\beta}$.

(25) Given the equation $m^2x^2+2mnx+n^2+1=0$ where n and m are constants, show that the equation has no real roots for any value of m and n.

Given that the common root of the equations $x^2-5x+2k=0$ and $x^2-2x-k=0$ is a where $k\neq 0$. Find k and a.
$(k=3,\ a=3)$

(b) SYMMETRICAL EQUATIONS

These are equations whose coefficients are symmetrical ie coefficients are the same when read from either side of the equation eg $ax^4 + bx^3 + cx^2 + bx + c = 0$ where a, b and c are constants.

Examples

① If $P = x + \frac{1}{x}$, express $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$ and $x^4 + \frac{1}{x^4}$ in terms of P.

Soln

$$P = x + \frac{1}{x}$$

$$P^2 = \left(x + \frac{1}{x}\right)^2$$

$$= x^2 + \frac{1}{x^2} + 2$$

$$\therefore \; x^2 + \frac{1}{x^2} = P^2 - 2$$

$$P^3 = \left(x + \frac{1}{x}\right)^3$$

$$= x^3 + \frac{1}{x^3} + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2$$

$$= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

$$= x^3 + \frac{1}{x} + 3P$$

$$\therefore \; x^3 + \frac{1}{x^3} = P(P^2 - 3)$$

$$P^4 = \left(x + \frac{1}{x}\right)^4$$

$$= x^4 + \frac{1}{x^4} + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3$$

$$= x^4 + \frac{1}{x^4} + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$= x^4 + \frac{1}{x^4} + 4(P^2 - 2) + 6$$

$$\therefore \; x^4 + \frac{1}{x^4} = P^4 - 4P^2 + 2$$

② Use the substitution $y = x + \frac{1}{x}$ to solve the equation $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

Soln

Divide all through by x^2

$$2x^2 + \frac{2}{x^2} - 9x - \frac{9}{x} + 14 = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$$

But $x^2 + \frac{1}{x^2} = y^2 - 2$

$\Rightarrow 2(y^2 - 2) - 9y + 14 = 0$

$y = 2$ or $y = 5/2$

when $y = 2$, $2 = x + \frac{1}{x}$; $x = 1$

when $y = 5/2$, $\frac{5}{2} = x + \frac{1}{x}$; $x = 2$ or $x = \frac{1}{2}$

$\therefore \{x : x = \frac{1}{2}, 1, 2\}$

Exercise

① Solve the equation;

$(x^2 - 2x)^2 + 24 = 11(x^2 - 2x)$

$\{x : x = -2, -1, 3, 4\}$

② Use the substitution $y = x + \frac{1}{x}$ to solve the equations below.

(i) $5x^4 - 16x^3 - 42x^2 - 16x + 5 = 0$

(ii) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0$

(iii) $x^4 + 3x^3 + 2x^2 + 3x + 1 = 0$

(iv) $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$

③ Using the substitution $y = x + \frac{4}{x}$, solve the equation given by;

$x^4 - x^3 - 12x^2 - 4x + 16 = 0$

④ Using the substitition $y = x^2 - 4x$, Solve the equation;

$2x^4 - 16x^3 + 77x^2 - 180x + 63 = 0$

⑤ Prove that if $y + 1 = x + \frac{1}{x}$ then $\frac{(x^2 - x + 1)^2}{x(x-1)^2} = \frac{y^2}{y-1}$ hence solve the equation $(x^2 - x + 1)^2 - 4x(x-1)^2 = 0$

(C) SIMULTANEOUS EQUATIONS

(1) Linear equations

Linear simultaneous equations can be solved using elimination and back substitution method, row reduction to Echelon form, Matrix method, cramer's rule or graphical method.

Elimination and back substitution method

In this method, one or more variable(s) is(are) eliminated until only one variable remains.

The value of this variable is then obtained and substituted in any of the equations to obtain the other variable(s)

Graphical method

In this method, graphs of the linear equations are drawn on the same axes. The values of the variables are then obtained as the coordinates of the point of intersection of the graphs drawn.

Row reduction to Echelon form

In this method, corresponding coefficients of the unknown variables are extracted and arranged in a 3×3 matrix form. The matrix is operated upon until when a triangle of zeroes is formed below the major diagonal. The values of the variables are then obtained.

Matrix method

In this method, corresponding coefficients of the unknown variables are extracted and arranged in a square matrix while the constants are arranged in a column matrix. The matrices are arranged in

form of an equation.
The equation is multiplied through by the inverse of the square matrix and the values of the variables are then obtained.

Examples

① Solve the simultaneous equations below:

(a) $3x + 2y = 2$
$4x - y = 10$

(b) $2x + 3y - 4z = 1$
$3x - y - 2z = 4$
$4x - 7y - 6z = -7$

Soln

(a) Using elimination method

$$3x + 2y = 2 \quad --- \quad (i)$$
$$\times 2 \quad 4x - y = 10 \quad --- \quad (ii)$$
$$11x = 22$$
$$x = 2$$

Substitute for x in (ii)

$$y = 4(2) - 10$$
$$= -2$$

$\therefore$ $x = 2$ and $y = -2$

Using matrix method

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} -3-8 & -2+2 \\ -12+12 & -8-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2-20 \\ -8+30 \end{pmatrix}$$

$$\begin{pmatrix} -11 & 0 \\ 0 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -22 \\ 22 \end{pmatrix}$$

$$x = \frac{-22}{-11}, \quad y = \frac{22}{-11}$$

$\therefore$ $x = 2$ and $y = -2$

Using substitution method

$3x + 2y = 2$ ------------ (I)

$4x - y = 10$ ------------ (II)

from (II) $y = 4x - 10$

Substitute for y in (I)

$3x + 2(4x - 10) = 2$

$11x = 22$

$x = 2$

$\Rightarrow y = 4(2) - 10$

$\therefore x = 2$ and $y = -2$

Using Cramer's rule

$3x + 2y = 2$

$4x - y = 10$

$$\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

Let $M = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$

$|M| = (-3 - 8) = -11$

$M_1 = \begin{pmatrix} 2 & 2 \\ 10 & -1 \end{pmatrix}$

$|M_1| = (-2 - 20) = -22$

$M_2 = \begin{pmatrix} 3 & 2 \\ 4 & 10 \end{pmatrix}$

$|M_2| = (30 - 8) = 22$

$$x = \frac{|M_1|}{|M|} = \frac{-22}{-11} = 2$$

$$y = \frac{|M_2|}{|M|} = \frac{22}{-11} = -2$$

$\therefore x = 2$ and $y = -2$

(b) Using substitution method:

$2x + 3y - 4z = 1$ ------- (I)

$3x - y - 2z = 4$ ------- (II)

$4x - 7y - 6z = -7$ ------- (III)

From (II), $y = 3x - 2z - 4$

Substitute for y in (I)

$$2x + 3(3x - 2z - 4) - 4z = 1$$
$$2x + 9x - 6z - 12 - 4z = 1$$
$$11x - 10z = 13$$
$$z = \frac{1}{10}(11x - 13)$$

Substitute for z and y in (III)

$$4x - 7(3x + \tfrac{1}{5}(11x-13) - 4 - \tfrac{3}{5}(11x-13) = -7$$
$$x = 3 \;;\; y = 1 \text{ and } z = 2$$
$$\therefore x = 3;\ y = 1 \text{ and } z = 2$$

Using cramer's rule

$$2x + 3y - 4z = 1$$
$$3x - y - 2z = 4$$
$$4x - 7y - 6z = -7$$

$$\begin{pmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$$

let $M = \begin{pmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{pmatrix}$

$$\det M = 2\begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 4 & -6 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ 4 & -7 \end{vmatrix}$$
$$= 82$$

$$M_1 = \begin{pmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ 7 & -7 & -6 \end{pmatrix}$$

$$\det M_1 = 1\begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3\begin{vmatrix} 4 & -2 \\ -7 & -6 \end{vmatrix} - 4\begin{vmatrix} 4 & -1 \\ -7 & -7 \end{vmatrix}$$
$$= 246$$

$$\Rightarrow x = \frac{246}{82} = 3$$

$$M_2 = \begin{pmatrix} 2 & 1 & -4 \\ 3 & 4 & -2 \\ 4 & -7 & -6 \end{pmatrix}$$

$$\det M_2 = 2\begin{vmatrix} 4 & -2 \\ -7 & -6 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 4 & -6 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 4 & -7 \end{vmatrix}$$
$$= 82$$

$$\Rightarrow y = \frac{82}{82} = 1$$

$$* \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

$$M_3 = \begin{pmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ 4 & -7 & -7 \end{pmatrix}$$

$$\det M_3 = 2\begin{vmatrix} -1 & 4 \\ -7 & -7 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 4 & -7 \end{vmatrix} + 1\begin{vmatrix} 3 & -1 \\ 4 & -7 \end{vmatrix}$$

$$= 164$$

$$\Rightarrow Z = \frac{164}{82} = 2$$

$\therefore$ $x = 3$, $y = 1$ and $Z = 2$

Using row reduction to Echelon form

$$2x + 3y - 4Z = 1$$
$$3x - y - 2Z = 4$$
$$4x - 7y - 6Z = -7$$

$$\begin{pmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} 2 & 3 & -4 & 1 \\ 3 & -1 & -2 & 4 \\ 4 & -7 & -6 & -7 \end{array}\right) \xrightarrow[\substack{r_2 \to 4r_2 - 3r_3 \\ r_3 \to r_3 - 2r_1}]{r_1 \to r_1} \left(\begin{array}{ccc:c} 2 & 3 & -4 & 1 \\ 0 & 17 & 10 & 37 \\ 0 & -13 & 2 & -9 \end{array}\right)$$

$$\left(\begin{array}{ccc:c} 2 & 3 & -4 & 1 \\ 0 & 17 & 10 & 37 \\ 0 & -13 & 2 & -9 \end{array}\right) \xrightarrow[\substack{r_2 \to r_2 \\ r_3 \to 17r_3 + 13r_2}]{r_1 \to r_1} \left(\begin{array}{ccc:c} 2 & 3 & -4 & 1 \\ 0 & 17 & 10 & 37 \\ 0 & 0 & 164 & 328 \end{array}\right)$$

$$\Rightarrow 164Z = 328$$
$$Z = 2$$
$$17y + 10z = 37$$
$$17y = 37 - 20 = 17$$
$$y = 1$$
$$2x + 3y - 4Z = 1$$
$$2x = 1 - 3(1) + 4(2) = 6$$
$$x = 3$$

$\therefore$ $x = 3$, $y = 1$ and $Z = 2$

Using matrix method

The method involves extracting out the corresponding coefficients and arranging them in a 3×3 matrix whose adjoint (adjunct) matrix is then obtained.

The adjoint matrix is obtained by replacing each element of the 3×3 matrix by its cofactor.

NB:

A cofactor of an element of a 3×3 order matrix is the 2×2 determinant obtained by deleting the row and column containing the element and multiplying it by +1 or −1 according to the pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Considering the above example;

$$2x + 3y - 4z = 1$$
$$3x - y - 2z = 4$$
$$4x - 7y - 6z = -7$$

$$\begin{pmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$$

let $M = \begin{pmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{pmatrix}$

$\Rightarrow \text{adj} M = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

NB: $M \,\text{adj} M = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} = \Delta I$

$\det M = \det M^T$, where M^T is the transpose matrix formed by substituting the rows with columns. eg. if $M = ($[illegible]$)$, $M^T = ($[illegible]$)$

* $M^{-1} = \frac{1}{\Delta} \text{adj} M$

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