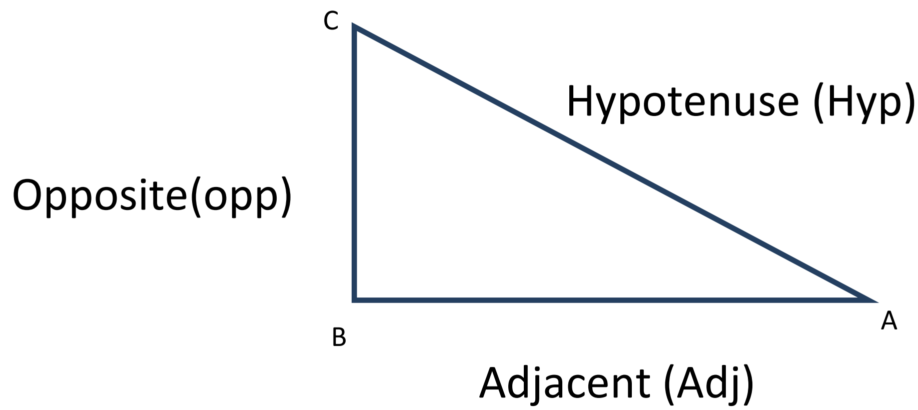
Ratios (sines, cosines and tangents) of angles less than 90°

Triangle ABC is right angled at B



Sin A = $\frac{opp}{Hyp} = \frac{BC}{AC}$

Cos A = $\frac{Adj}{Hyp} = \frac{AB}{AC}$

Tan A = $\frac{Opp}{Adj} = \frac{BC}{AB} = \frac{Sin\ A}{Cos\ A}$

We can remember the ratios by using **SOH-CAH-TOA**which can be thought of as the first letters of the words representing the sides of the triangle.

Reading tables

Using the 4-figured tables, find the sine, cosine and tangent of:

i) 60^0 ii) 40^0 iii) 55^0iv) 73^0 v) 50^0 vi) 30^0

For any right angled triangle, the cosine of an angle is equal to the sine of its complementary angle.

I.e. Cos A = Sin(90-A)

Sine A = Cos(90-A)

Note that Cos 50^0 = Sin 40^0

Cos 40^0 = Sin 50^0

<u>The special angles; 0°, 30°, 45°, 60°, 90°</u>

The angles have exact ratios

<u>The ratios of 0°</u>

Angle is 0°, when opposite = zero and adjacent = Hypotenuse.

Cos 0° = 1

Sin 0° = 0

Tan 0° = 0

<u>The ratios of 90°</u>

Angle is 90°, when adjacent = zero and opposite = Hypotenuse

Cos 90° = 0

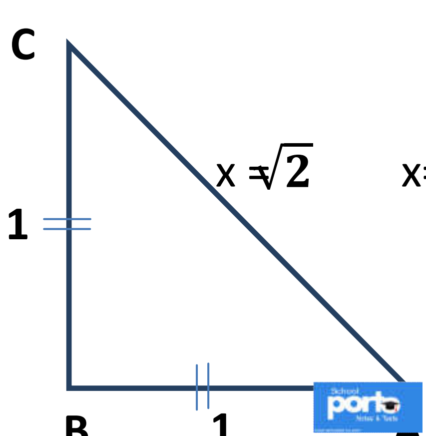
Sin 90° = 1

Tan 90° $\rightarrow \infty$

<u>The ratios of 45°</u>

$x^2 = 1^2 + 1^2$

$x^2 = 2$



$x = \sqrt{2}$

Triangle ABC is an isosceles triangle which angle B = 90°, and angle A = angle C = 45°

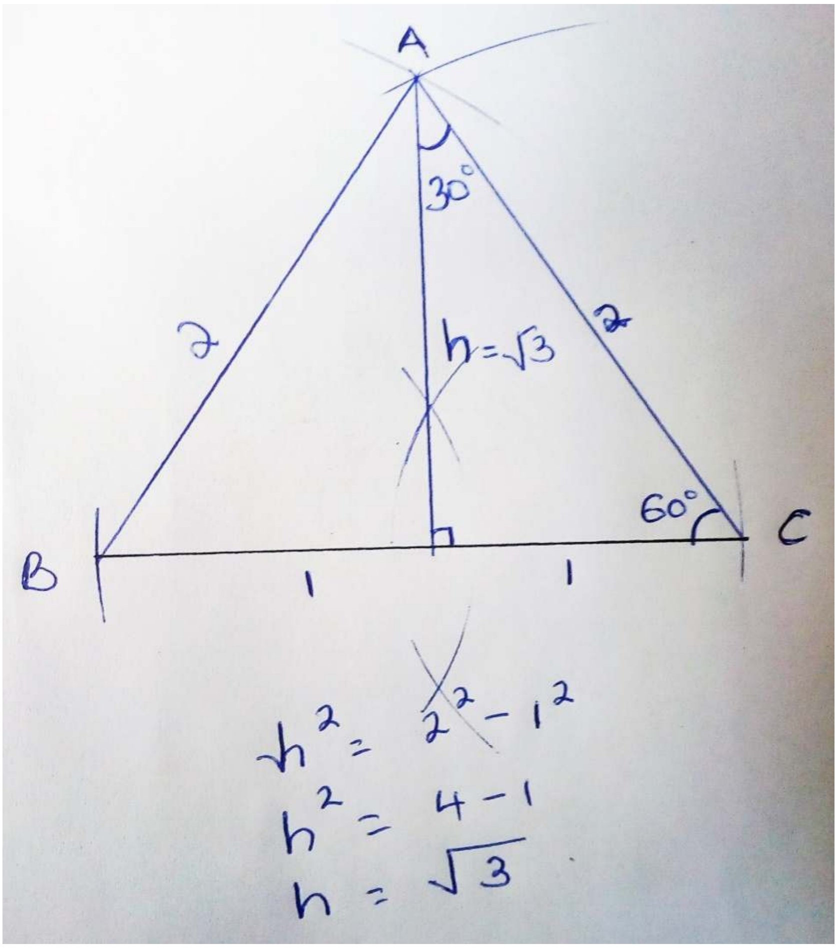
Cos 45° = $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Sin 45° = $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Tan 45° = 1

<u>The ratios of 60° and 30°</u>

Use an equilateral triangle of sides equal to 2 units with a perpendicular bisector of BC from A. The perpendicular bisector of BC also bisects angle A.



Ratios of 60°

Cos 60° = $\frac{1}{2}$

Sin 60° = $\frac{\sqrt{3}}{2}$

Tan 60° = $\sqrt{3}$

Ratios of 30°

Cos 30° = $\frac{\sqrt{3}}{2}$

Sin 30° = $\frac{1}{2}$

Tan 30° = $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Angle A	0°	30°	45°	60°	90°
Cos A	1	$\sqrt{3}/_2$	$\sqrt{2}/_2$	$^1/_2$	0
Sin A	0	$^1/_2$	$\sqrt{2}/_2$	$\sqrt{3}/_2$	1
Tan A	0	$^1/_3$	1	$\sqrt{3}$	∞

<u>Example:</u>

Without using tables or calculators, evaluate leaving your answers in rational surd form.

a) $\frac{\cos\ 60°+\sin 60°}{Tan\ 60°}$

$=\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{\sqrt{3}}$

$=\frac{1+\sqrt{3}}{2\sqrt{3}}$

$=\frac{(1+\sqrt{3})\sqrt{3}}{2\times 3}$

$=\frac{\sqrt{3}+3}{6}$

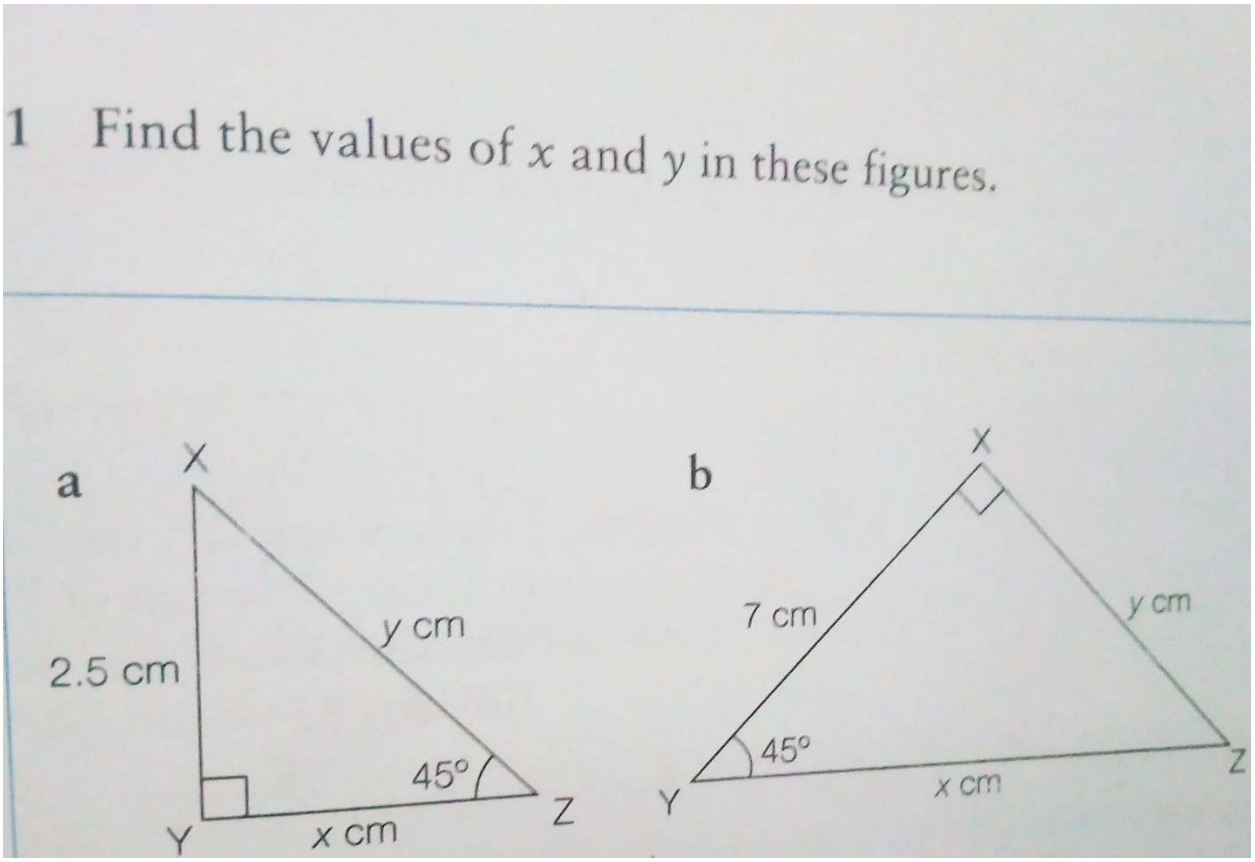
b) $\frac{Cos\ 60°\ Sin\ 60°}{Sin\ 90°-Tan\ 30°}$

c) $\text{Tan}^2\,60° + 3\text{Cos}^2\,45°$

d) Sin(45°) Cos(45°)

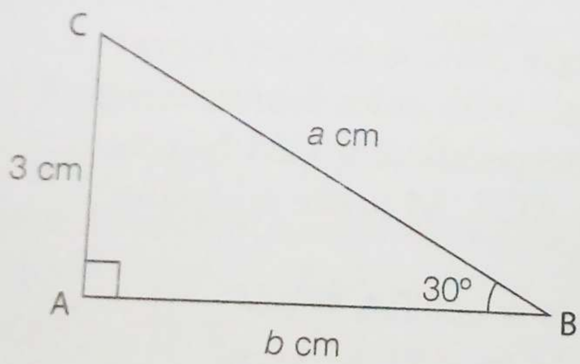
Note that: $\text{Tan}^2 60° = (\text{Tan } 60)^2$

Exercise

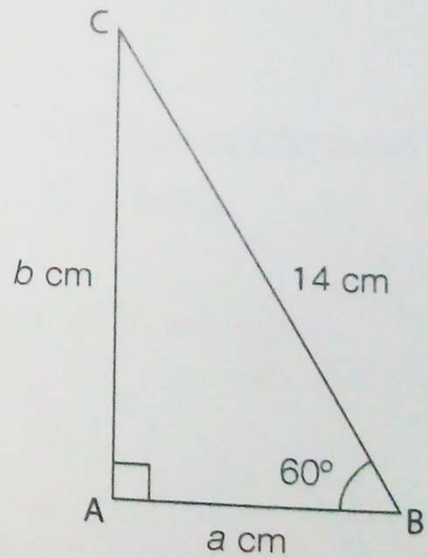

1 Find the values of x and y in these figures.

2 Find the values of a and b in these figures.

a

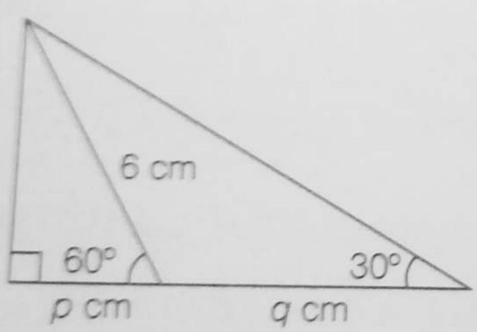


b

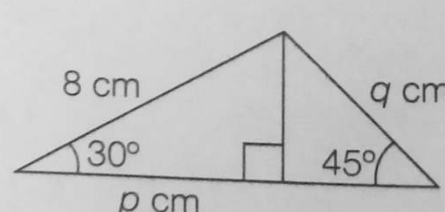


3 Find the values of p and q in these figures.

a



b



4 a Write down the values of sin 67°, cos 67° and tan 67°.

b Show that (i) $\frac{\sin 67^\circ}{\cos 67^\circ} = \tan 67^\circ$

(ii) $\sin^2 67^\circ + \cos^2 67^\circ = 1$

5 a Find the values of sin 55°, cos 55°, sin 35° and cos 35°.

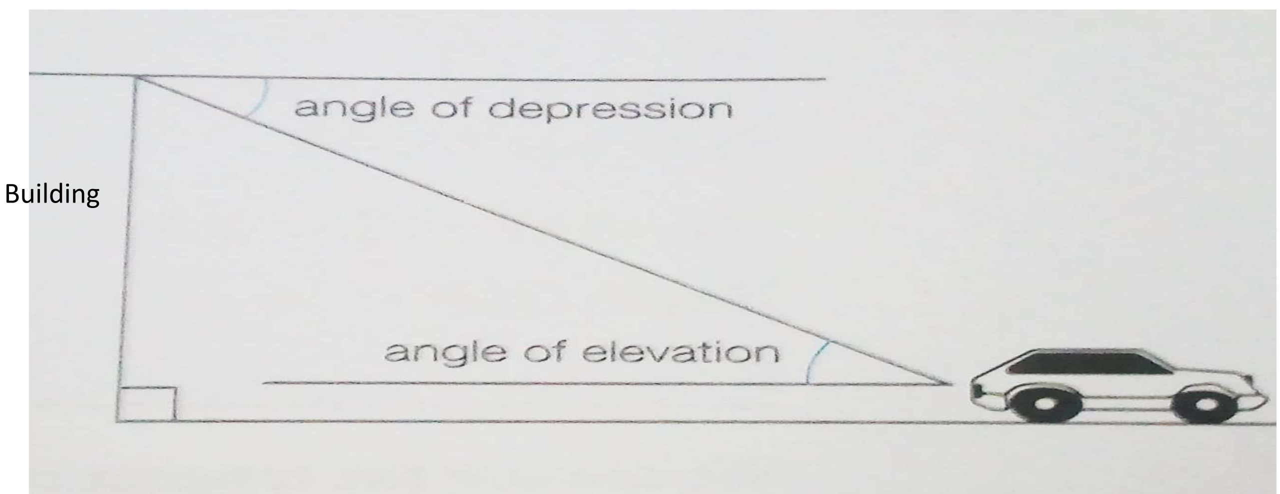
b Show that (i) $\sin 55^\circ = \cos(90^\circ - 55^\circ)$

(ii) $\sin^2 35^\circ + \cos^2 35^\circ = 1$

(iii) $\cos 35^\circ = \sin(90^\circ - 35^\circ)$

Angle of Elevation and Angle of Depression

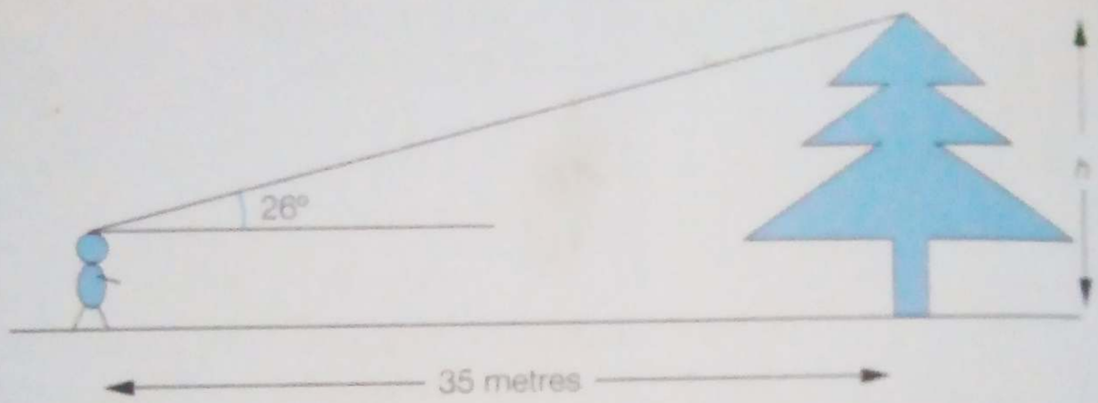
An observer in a car at some distance sees a bird on top of a tall Building as shown in the diagram below.



Note that the angle at which the observer sees the bird is the Same as the angle at which the bird sees the observer. Hence angle of depression is equal to angle of elevation.

Example

Eric is standing 35 m from a tree. He measures the angle of elevation of the top of the tree and finds that it is 26°. Eric is 1.6 m tall. How tall is the tree?



The top of the tree is $(h - 1.6)$ metres taller than Eric.

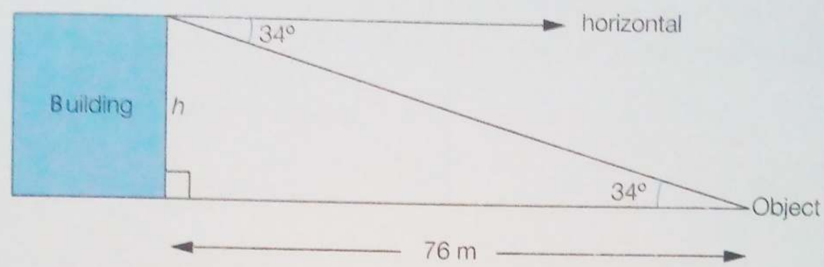
$$\tan 26^\circ = \frac{\text{opp}}{\text{adj}} = \frac{h - 1.6}{35}$$

$$h - 1.6 = 35 \tan 26^\circ = 17.1 \text{ m}$$
$$h = 1.6 + 17.1 = 18.7 \text{ m}$$

The tree is 18.7 m tall.

Example

The angle of depression of an object on the ground from the top of a building is 34°. The horizontal distance from the object on the ground to the base of the building is 76 metres. What is the height of the building?



If the height of the building is h metres,

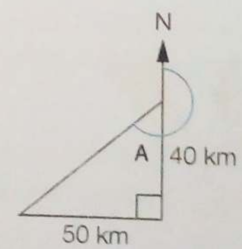
$\tan\ 34° = \frac{\text{opp}}{\text{hyp}} = \frac{h}{76}$

and $h = 76 \tan 34° = 51.3$ metres.

Example A ship sails 40 km south from a port and then 50 km west. What is its bearing from the port?

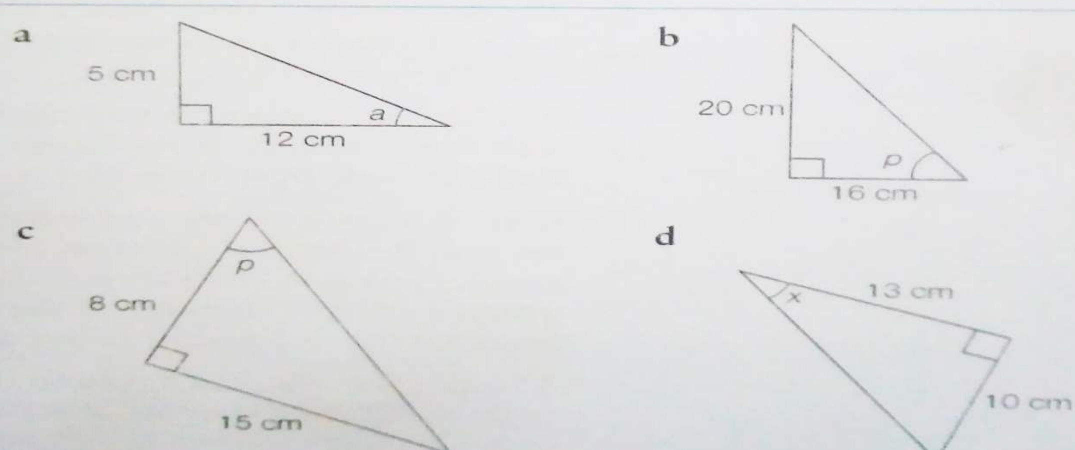
$\tan\ A = \frac{50}{40}$ so $A = 51.3°$

The bearing is $180° + A = 231°$



Exercise

1 Find the angles marked with letters.



3 Ekanya, who is 1.7 m tall, stands 20 m from a vertical tree and finds that the angle of elevation of the top of the tree is 23°. Find the height of the tree, assuming that the tree is perpendicular to the ground.

4 The angle of elevation of the top of a flagpole is 54° from a point on the ground 10 m from the base of pole. Find the height of the flagpole.

5 From the top of a vertical cliff 65 metres high, the angle of depression of a boat is 28°. Find the distance of the boat from the foot of the cliff.

6 The distance of a boat from a vertical cliff is 1500 metres. The angle of depression of the boat from the top of the cliff is 6°. Find the height of the cliff.

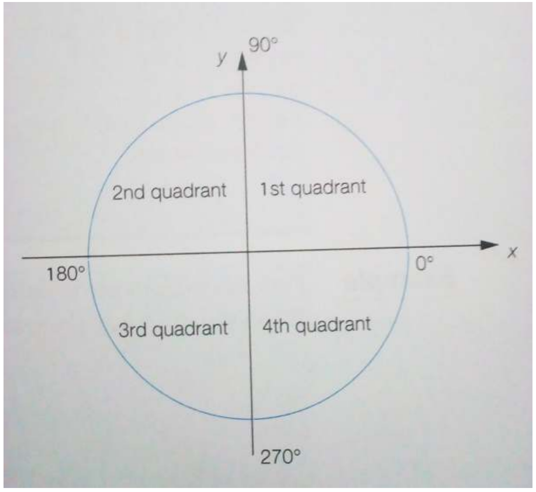
7 A ladder is placed with its foot 6 metres from the bottom of a wall 9 metres high. If the ladder reaches the top of the wall, find the angle that the ladder makes with the ground.

8 A tree is 12 metres tall, and, from a point level with the base of the tree, the angle of elevation of the top of the tree is 23°. From another point, in line with the first point and the base of the tree, the angle of elevation of the top of the tree is 18°. How far apart are the two points? (There are two possible answers.)

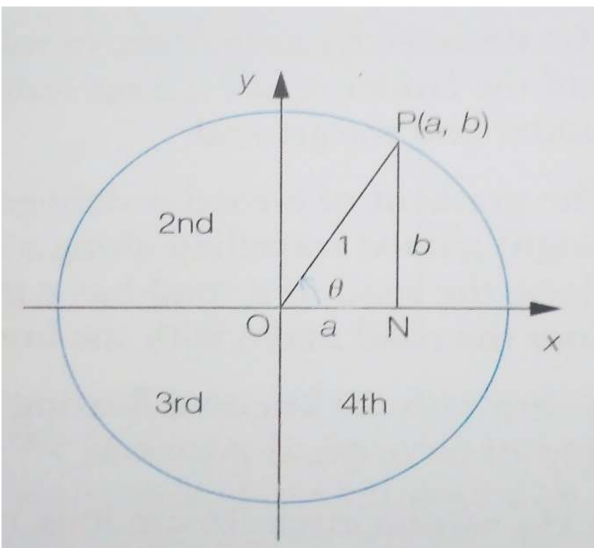
9 A ladder rests against a wall in such a way that it makes an angle of 42° with the wall and its foot is 7 metres from the wall. Calculate the height reached by the top of the ladder.

<u>Angles greater than 90°</u>

Consider a unit circle,



In i) 0°<Ө <90°, P is the first quadrant.

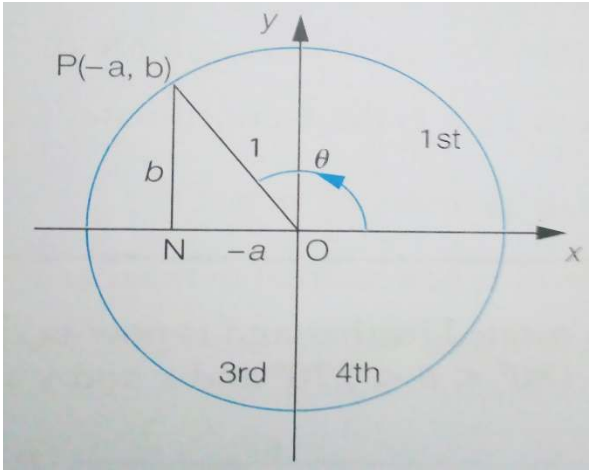


ON = 1 x Cos Ө = Cos Ө

NP = 1 x Sin Ө = Sin Ө.

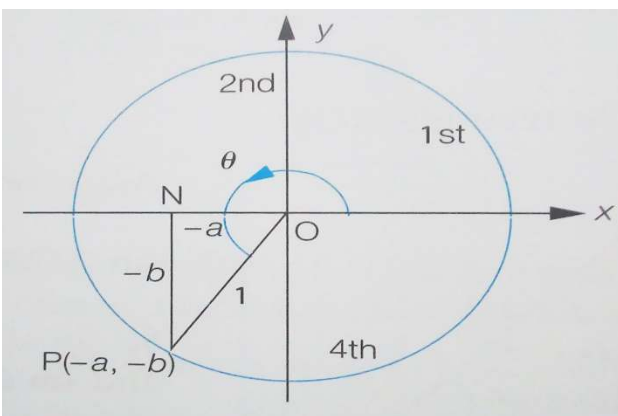
and hence the coordinates of P(Cos Ө, Sin Ө). Both the Cosine and the sine are positive. Hence Tangent is also positive.

In ii) 90°<Ɵ< 180°,



Ɵ is in the second quadrant so that x-coordinate(a) is negative while the y-coordinate (b) is positive. Therefore for obtuse angles, sines (+ve) but cosines and tangents are (-ve)

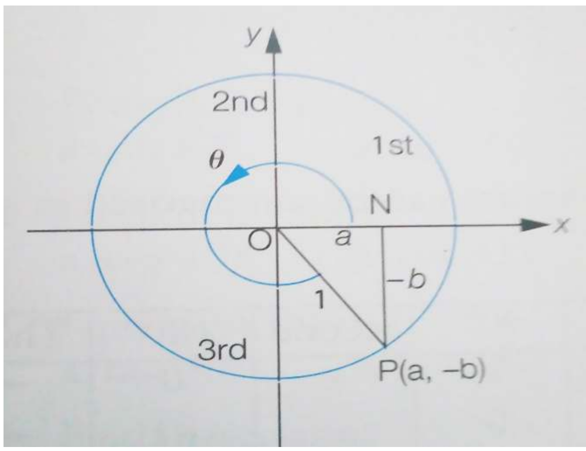
In iii) 180°<Ɵ<270°,



Ɵ is in the third quadrant. The x- and y-coordinates are negative, therefore

sines(-ve)and cosines(-ve) while the tangents (+ve).

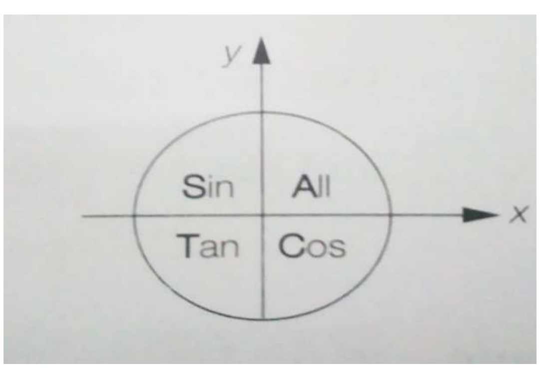
In iv) 270°<Ө<360



Ө is in the fourth quadrant. Then x is positive and y negative.

Therefore the cosines (+ve) while the sines and tangents (-ve).

In summary, sines, cosines and tangents are all positive in the 1st quadrant.Sines are positive in the 2nd, tangents in the 3rd and cosine in the 4th.



Note

$\sin 150° = \sin(180 - 150)$
$= \sin 30°$

$\cos 150° = -\cos(180 - 150°)$
$= -\cos 30°$

$\cos 240° = -\cos(240 - 180°$
$= -\cos 60°$

Exercise:

Use four figured tables to fined sines, cosines, and tangents of the following angles.

a) 125°
b) 282°
c) 180°
d) 196°
e) 305°
f) 25°
g) 135°
h) 300°
i) 250°

WAVES (TRIGONOMETRIC FUNCTIONS)

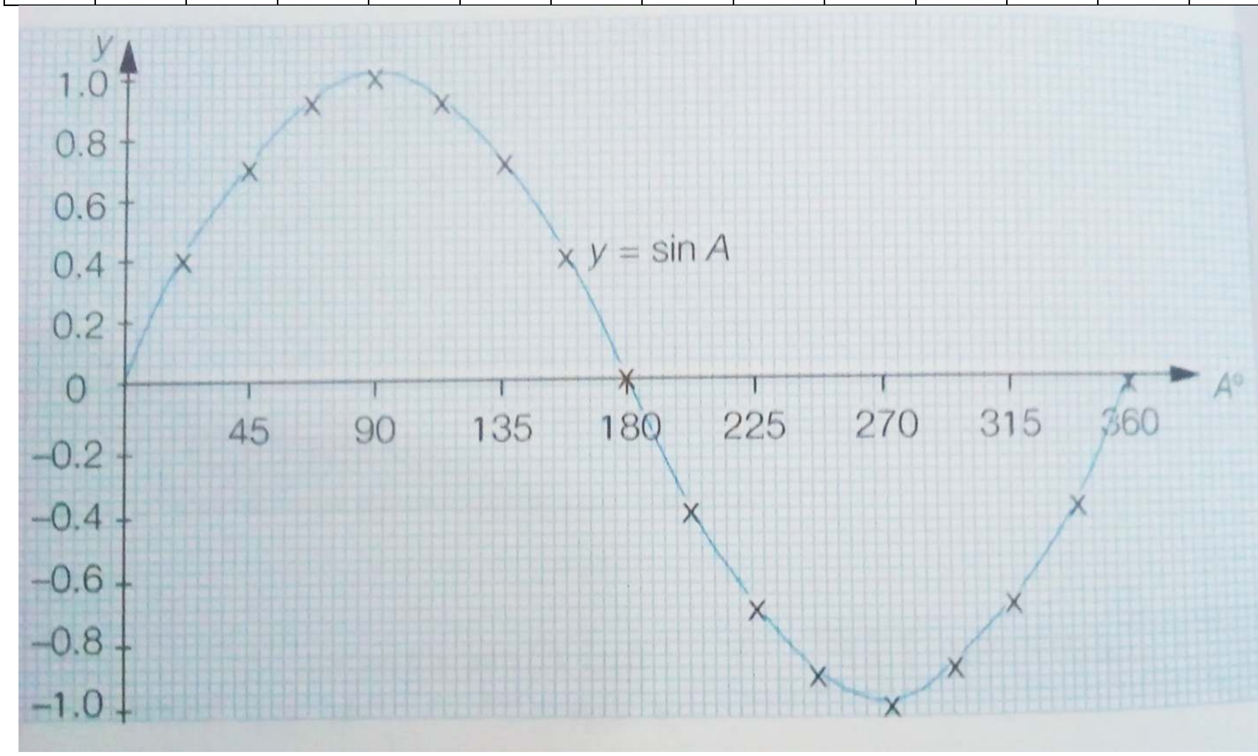
Sine functions

Plot the graph of sinϴ in the range $0 \leq \Theta \leq 360°$

Solution:

Let $y = \sin \Theta$.

Ɵ	0°	30°	60°	90°	120°	150°	180	210°	240°	270°	300	330	360
sinƟ	0.00	0.05	0.87	1.00	0.87	0.50	0.00	- 0.50	- 0.87	- 1.00	- 0.87	-0.5	0.00



Give the values of Ɵ for which sin Ɵ = 0.5 for 0°≤Ɵ≤360°

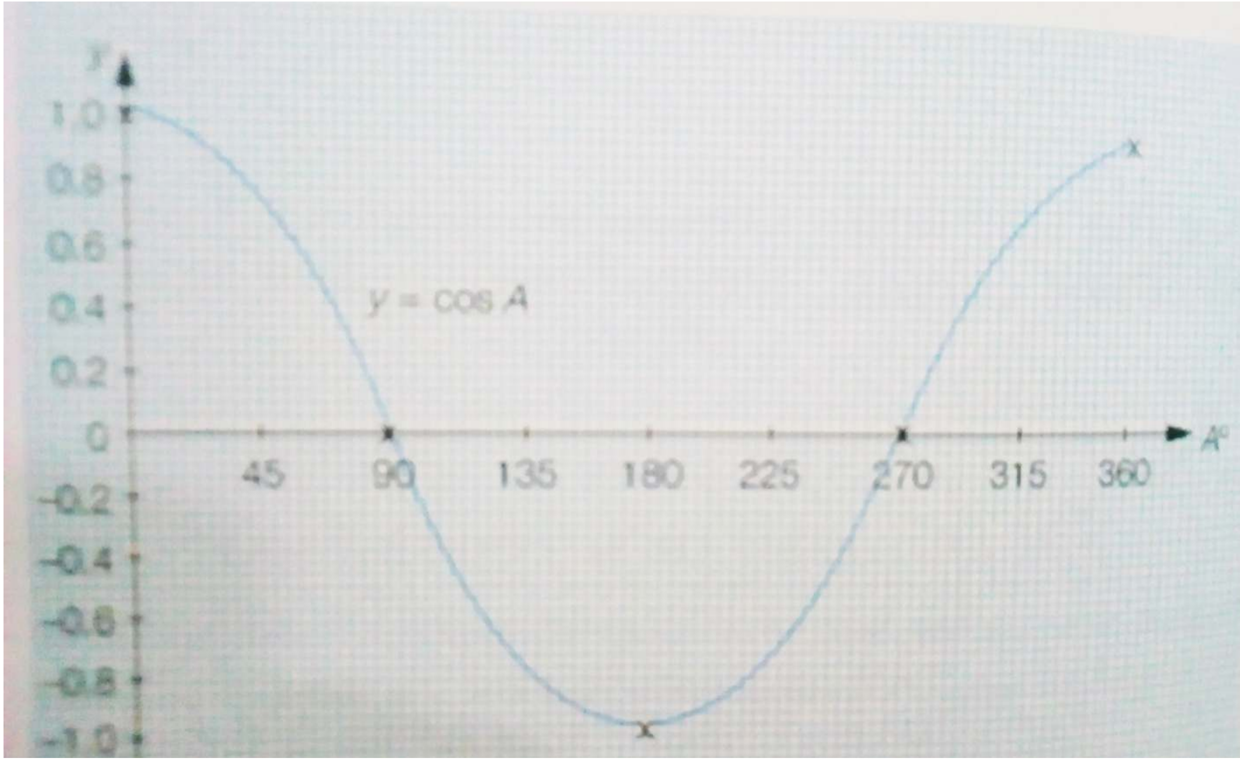
<u>For Cosine function</u>

The plot the graph of Cos Ɵ for which 0°≤Ɵ≤360°

<u>Solution</u>

Let y = Cos Ɵ

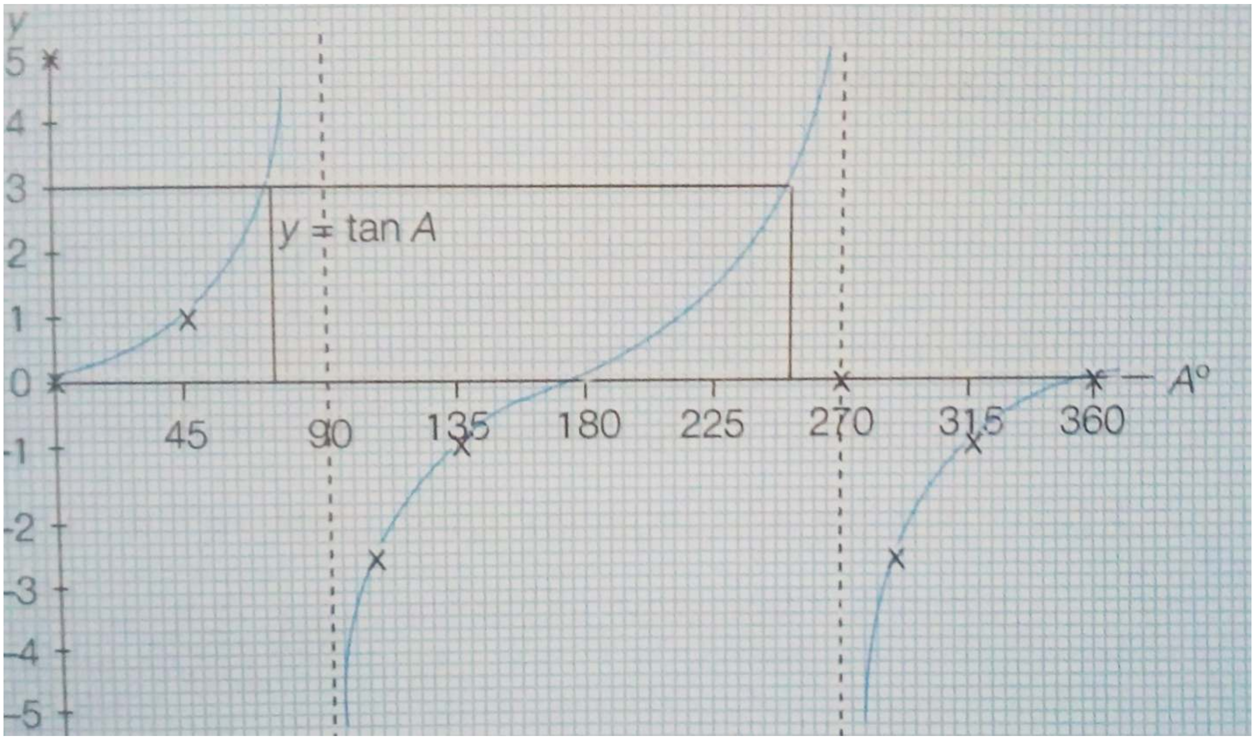
Ɵ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
CosƟ	1.00	0.87	0.5	0	-0.5	- 0.87	- 1.00	- 0.87	-0.5	0	0.5	0.87	1



<u>The Tangent function.</u>

Plot the graph of Tan Ɵ for which 0°≤Ɵ≤360°

Ɵ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
TanƟ	0	0.27	0.58	1	1.73	3.73	∞	-3.73	-1.73	-1	-0.58	-0.27	0
Ɵ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°	
TanƟ	o.27	0.58	1	1.73	3.73	∞	-3.73	-1.75	-1	-0.58	-0.27	0	



<u>Exercise:</u>

1) Find from your graph of y = Tan Θ the values of (y = tan Θ) which satisfies the following equations given the range 0°≤Θ≤270°

i) Tan x = 3

ii) Tan x = 0.65

iii) Tan x = -2.5

2) On the same pair of axes, draw the graphs of Tan Θ and Cos Θ for 0°≤Θ≤180°.

State the values of Θ for which tan Θ = Cos Θ

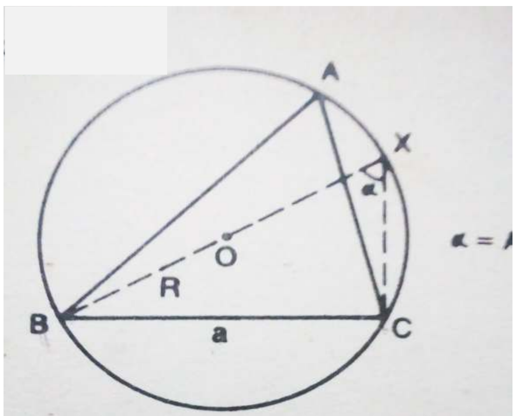
3) On the same pair of axes, draw the graphs of Sin Θ and Cos Θ for which 0°≤Θ≤180°.

The Sin rule.

Both the Sine and the Cosine rules are used to find lengths and angles in any triangle, while the Sine, Cosine and Tangent ratios are only used on right angled triangles.

In a triangle, the small letters are used for the sides while the capital letters are for the angles.

In the figure below, O is the center of the circumcircle of a triangle ABC with diameter BOX and R the radius.



Angle BCX is a right angle (angle in a semi-circle)
Angle BAC = BXC (angles in the same segment)

In triangle BXC

$Sin \ \propto = \frac{a}{2R} = Sin\ A$

$\frac{a}{Sin\ A}$ =2R

When the same procedure is done for B and C, we obtain

$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$= 2R

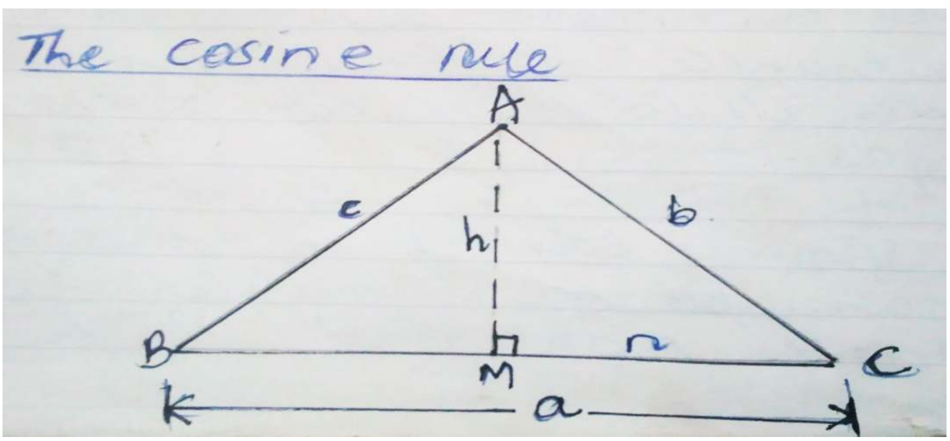
∴This is the Sine rule for the triangle ABC.

EXERCISE 4d

In questions 1–12 solve $\triangle ABC$ from the data, using the sine rule. Give side lengths to 2S and angles to the nearest degree.

1 $a = 12{\cdot}4$ cm, $B = 37°$, $C = 84°$.
2 $a = 0{\cdot}92$ cm, $B = 66°$, $C = 42°$.
3 $b = 4{\cdot}8$ km, $C = 70°$, $A = 69°$.
4 $b = 8{\cdot}3$ cm, $C = 22°$, $A = 35°$.
5 $c = 1{\cdot}64$ m, $A = 57°$, $B = 49°$.
6 $A = 43°$, $a = 4{\cdot}9$ cm, $b = 6{\cdot}2$ cm
7 $C = 71°$, $a = 7{\cdot}3$ m, $c = 8{\cdot}4$ m
8 $B = 108°$, $b = 5{\cdot}6$ cm, $c = 3{\cdot}8$ cm
9 $B = 27°$, $a = 6{\cdot}7$ cm, $b = 3{\cdot}8$ cm
10 $A = 63°$, $a = 7{\cdot}3$ cm, $c = 8{\cdot}2$ cm
11 $B = 131°$, $a = 10{\cdot}3$ m, $b = 6{\cdot}9$ m
12 $C = 58°$, $a = 12{\cdot}2$ km, $c = 10{\cdot}9$ km

The Cosine rule



For triangle ABM

$AB^2 = AC^2 – AM^2$

$C^2 = (a - n)^2 + h^2$

$= a^2 – 2an + n^2 + h^2$...........(1)

For triangle AMC

$MC^2 = AC^2 - AM^2$

$n^2 = b^2 - h^2$...............................(2)

Combining the equations (1) and (2)

$c^2 = a^2 - 2an + (b^2 - h^2) + h^2$

$\quad = a^2 - 2an^2 + b^2$..........................(3)

Also from triangle AMC

n = b $Cos\ C$.....................................(4)

Combining (3) and (4).

$c^2 = a^2 - 2a(b\ Cos\ C) + b^2$

$c^2 = a^2 + b^2 - 2ab\ Cos\ C$

<u>N.B</u> The same can be done for b and B, and a and A.

i.e. $a^2 = b^2 + c^2 - 2ac\ Cos\ A$

$\quad b^2 = a^2 + c^2 - 2ac\ Cos\ B$

∴ This is the Cosine rule of a triangle ABC

EXERCISE 4e

In questions 1–5, solve the triangles by using the cosine rule once and then the sine rule.

1 $B = 67°$, $a = 7{\cdot}1$ cm, $c = 5{\cdot}2$ cm
2 $A = 58°$, $b = 14$ cm, $c = 23$ cm
3 $C = 132°$, $a = 150$ m, $b = 120$ m
4 $A = 47°$, $b = 1{\cdot}42$ km, $c = 2{\cdot}51$ km
5 $B = 118°$, $c = 82$ cm, $a = 167$ cm

Stay Home, Stay safe.