TRIGONOMETRY

Trigonometry is a branch of mathematics that studies relationships involving lengths and angles of a triangle. It comes from two Greek words – *trigonom* (triangle) and *metron* (measure).

There is an enormous number of the uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance between land marks. Although it was first applied in spheres, it had a greater application to planes. Surveyors have used trigonometry for many centuries.

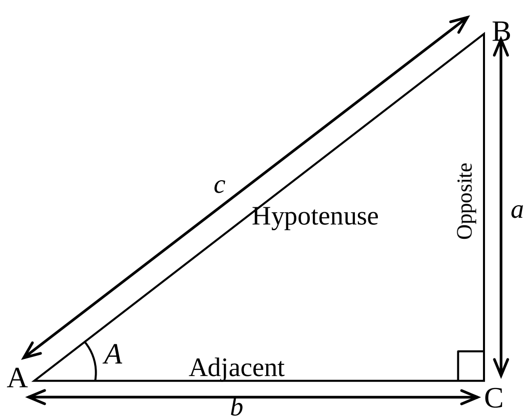
Within mathematics, it is used in calculus (perhaps its greatest application), linear algebra, and statistics.

Trigonometric tables were created over 2000 years ago for computation in astronomy.

A student is expected to be familiar with the definitions of trigonometric ratios for acute angles.

If one angle is 90° and one of the other angles is known, the third can be determined because the three angles of any triangle add up to 180°. The two acute angles therefore add up to 90° (complimentary angles).

Once the angles are known, the ratios of the sides are determined regardless of the overall size of the triangle. If the length of one side is known, the other two are determined. These ratios are given by the following trigonometric functions of known angle, *A*; where *a*, *b*, and *c* refer to the lengths of the sides accompanying the figure.



Sine function (sin)
This is the ratio of the opposite side of the triangle to its hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

Cosine function (cos)
This is the ratio of the adjacent side to the hypotenuse

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

Tangent function (tan)

This is the ratio of the opposite to the adjacent side.

$$\tan A = \frac{a}{b} = \frac{a}{c} \times \frac{c}{b}$$
$$= \left(\frac{a}{c}\right) \div \left(\frac{b}{c}\right)$$
$$= \frac{\sin A}{\cos A}$$
$$\tan A = \frac{\sin A}{\cos A}$$

The hypotenuse is the side opposite to the 90° angle. It is the longest side of a triangle and one of the sides adjacent to A.

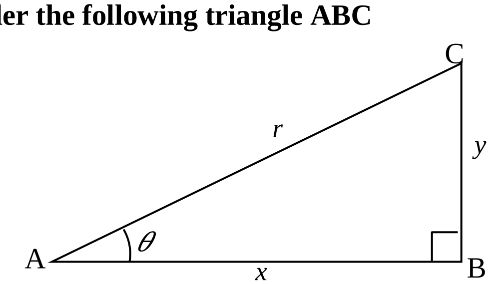
The term perpendicular and base are sometimes used for opposite and adjacent sides respectively.

Many people find it easy to remember what sides of the right angle are equal to sine, cosine, or tangent by memorising the mnemonic **SOH-CAH-TOA**.

The reciprocals of the functions are named cosecant (cosec), secant (sec) and cotangent (cot)

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{c}{a}$$
$$\sec A = \frac{1}{\cos A} = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{c}{b}$$
$$\cot A = \frac{1}{\tan} = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{\cos A}{\sin A} = \frac{b}{a}$$

Consider the following triangle ABC



$$\sin\theta = \frac{y}{r}\ ,\ \cos\theta = \frac{x}{r};\ \text{and}\ \tan\theta = \frac{y}{r}$$
$$y = r\sin\theta; \qquad x = r\cos\theta$$

Applying the Pythagoras' theorem to triangle ABC;

$\Rightarrow (r\cos\,\theta)^2 + (r\sin\,\theta)^2 = r^2$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\cos^2\theta + \sin^2\theta = 1$

$\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$ …………………………. (i)

$\boxed{\boldsymbol{\cos^2\theta + \sin^2\theta = 1}}$

Dividing equation (i) by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$1 + \tan^2\theta = \sec^2\theta$

$\boldsymbol{1 + \tan^2\theta = \sec^2\theta}$ …………………………….. (ii)

$\boxed{\boldsymbol{1 + \tan^2\theta = \sec^2\theta}}$

Dividing Eqn (i) by $\sin^2\theta$

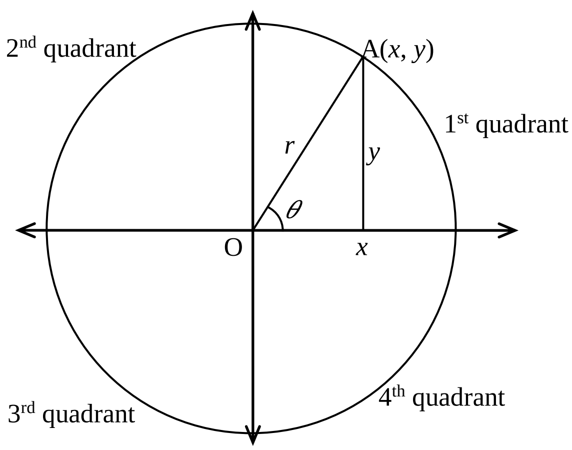
$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$\cot^2\theta + 1 = \text{cosec}^2\theta$

$\boldsymbol{1 + \cot^2\theta = \text{cosec}^2\theta}$ ……………..…………….. (iii)

$\boxed{\boldsymbol{1 + \cot^2\theta = \text{cosec}^2\theta}}$

Trigonometric Ratios for general angle



Angles measured from the *x*-axis in the anti-clockwise sense are termed as positive angles while those measured in the clockwise sense are negative angles.

When A is in the 1st quadrant, *x* and *y* are positive. When A is in the 2nd quadrant, *x* is negative and *y* is positive. When A is in the third quadrant, *x* and *y* are all negative. When A is in the 4th quadrant, *x* is positive and *y* is negative. *r* is taken to be positive for all positions of the line *OA*.

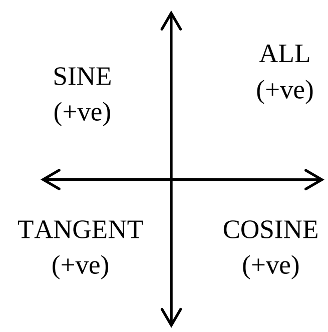
The trigonometrical ratios for angles *xOA* of any magnitude are defined precisely in the same way as for acute angles.

Thus $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ and $\tan\theta = \frac{y}{x}$

The appropriate signs are attached to *x* and *y* according to the position of point A. hence for angles in which OA lies in the 1st quadrant; since *x* and *y* and *r* are positive, the sine, cosine, and tangent will all be positive.

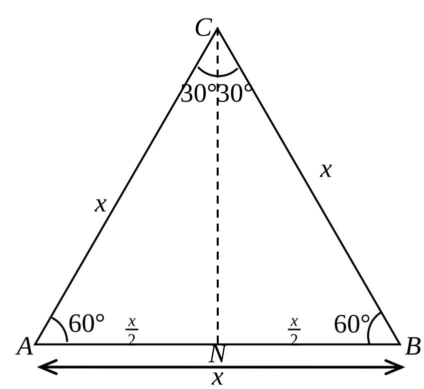
For angles in which OA lies in the 2nd quadrant, since *y* and *r* are positive and *x* negative, the sine is positive. Cosine and tangent are negative.

For angles in which OA is in the 3rd quadrant, sine and cosine are both negative but tangent is positive. In the 4th quadrant, sine and tangent are negative while cosine is positive. This is illustrated below.

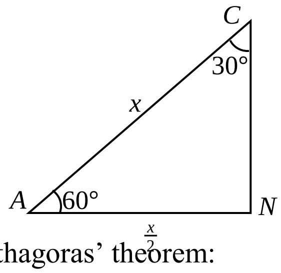


Trigonometric ratios of 30°, 45°, and 60°

Consider the equilateral triangle *ABC* of side *x*



Considering triangle CAN:



Applying the Pythagoras' theorem:

$$\left(\tfrac{x}{2}\right)^2 + (\overline{CN})^2 = x^2$$

$$\frac{x^2}{4} + (\overline{CN})^2 = x^2$$

$$(\overline{CN})^2 = x^2 - \frac{x^2}{4}$$

$$\overline{CN}^2 = \frac{3x^2}{4}$$

$$\overline{CN} = \frac{\sqrt{3}x}{2}$$

Using ***SOH-CAH-TOA***

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin\theta = \frac{\text{CN}}{\text{AC}}$$

$$\sin 60° = \frac{\frac{\sqrt{3}}{2}x}{x}$$

$$= \frac{\sqrt{3}}{2}$$

$$\sin 30° = \frac{x/2}{x} = \frac{1}{2}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 60° = \frac{\frac{x\sqrt{3}}{2}}{x} = \frac{\sqrt{3}}{2}$$

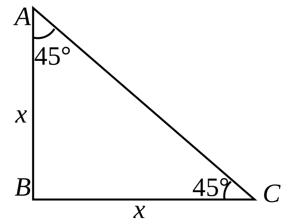
$$\cos 30° = \frac{\frac{\sqrt{3}}{2}x}{x} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\frac{x\sqrt{3}}{2}}{x/2} = \sqrt{3}$$

$$\tan 30 = \frac{x/2}{x\sqrt{3}/2}$$

$$\tan 30 = \frac{1}{\sqrt{3}}$$

Consider a right isosceles triangle with two sides of lengths *x* units.



Applying the Pythagoras' theorem on *ABC*:

$$x^2 + x^2 = AC^2$$

$$2x^2 = AC^2$$

$$AC = x\sqrt{2}$$

Applying *SOH-CAH-TOA*

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{AC}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\sin 45° = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\sin 45° = \frac{x}{x} = 1$$

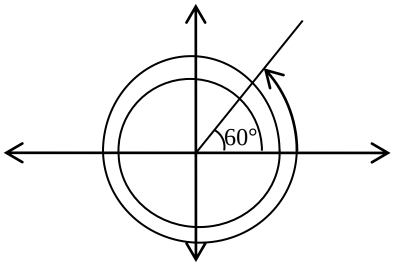
Example I

Write down the values of the following, leaving surds in your answers (*the calculator should not be used*).

(a) cos 780°
(b) sin 780°
(c) tan 780°
(d) sin 540°
(e) cos 540°
(f) cos 210°
(g) sin 150°
(h) sin(-270°)
(i) sin 225°
(j) sin 405°
(k) tan(-60°)

Solution

(a) cos 780.



cos 780° = cos 60°

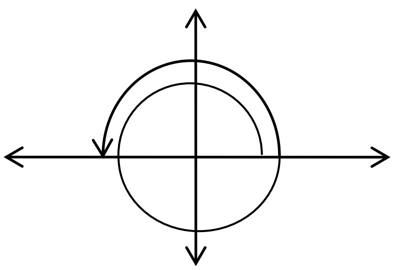
$= \frac{1}{2}$

sin 780° = sin 60°

$= \frac{\sqrt{3}}{2}$

tan 780° = tan 60° = $\sqrt{3}$

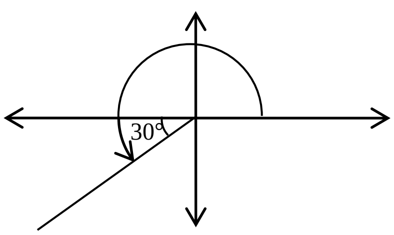
sin 540°



sin 540° = sin 180° = 0°

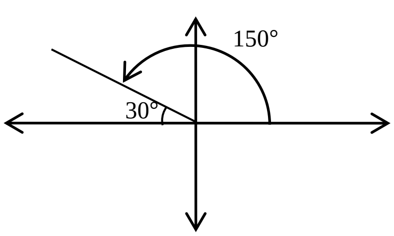
cos 540° = cos 180° = 0°

cos 210°



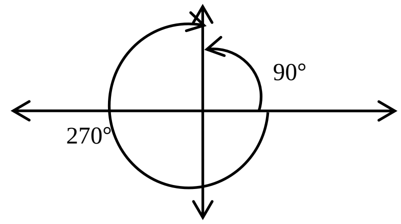
cos 210° = -cos 30° = $\frac{-\sqrt{3}}{2}$

sin 150°



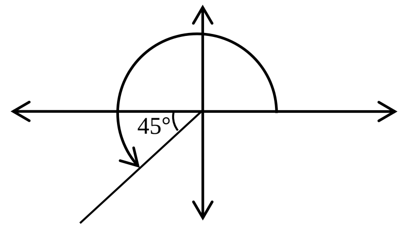
sin 150 = +sin 30 = $\frac{1}{2}$

sin -270°



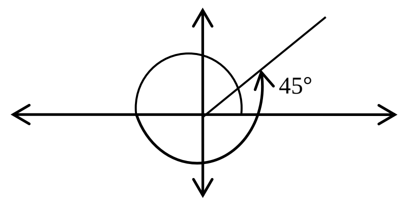
sin -270 = +sin 90° = 1

sin 225°



sin 225° = -sin 45° = $\frac{-1}{\sqrt{2}}$

sin 405°



sin 405° = sin 45° = $\frac{1}{\sqrt{2}}$

Trigonometric Curves

For any angle θ, a single value of sin θ or cos θ can be found. The same applies to tan θ unless when $\theta = \pm 90°$ and $\pm 270°$ for which the values of tan θ are not defined. Thus sin θ and cos θ are functions which are defined for all negative values of θ.

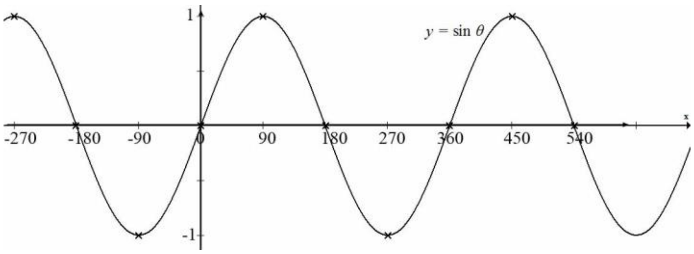
Tan θ is a function which is defined for all positive and negative values of θ except $\pm 90°$ and $\pm 270°$.

To draw the graphs of sinθ, cosθ and tanθ, we construct a table of values, giving ordered pairs of these functions and hence plot the graph.

Example

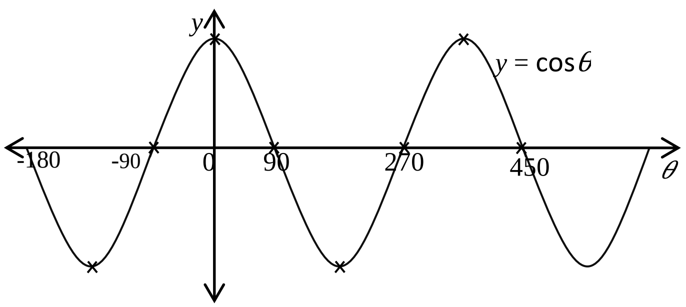
$y = \sin\theta$

θ	-270	-180	-90	0	90	180	270	360	450	540
$y = \sin\theta$	1	0	-1	0	1	0	-1	0	1	0



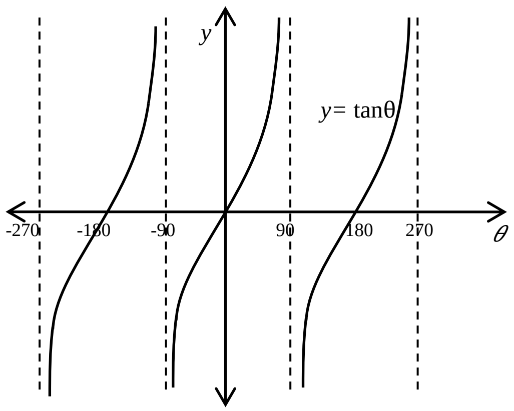
$y = \cos\theta$

θ	-180	-90	0	90	180	270	360	450
$y = \cos\theta$	1	0	1	0	-1	0	1	0



$y = \tan\theta$

θ	-270	-180	-90	0	90	180	270	360	450
$y = \tan\theta$	∞	0	∞	0	∞	0	∞	0	∞



From the graph of sin θ and cos θ, the maximum values of cos θ and sin θ are 1 and 1 respectively. The minimum value of cos θ and sin θ are -1 and -1 respectively.

The graphs for sinθ and cosθ repeat themselves at regular intervals of 360° while that of tanθ repeat itself at regular interval of 180°. These intervals are called periods. These trigonometric functions are examples of periodic functions.

Trigonometric Equations

Trigonometric equations differ from algebraic equations in that they often have unlimited number of solutions.

Example I

Solve the following equations for 0≤ θ ≤ 360°

(a) $\sin\theta = \frac{-1}{2}$

(b) $\sec\theta = 2$

(c) $\tan\theta = -\sqrt{3}$

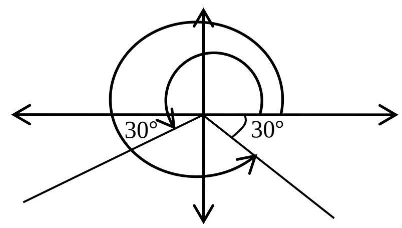
(d) $\sin^2\theta = \frac{1}{2}$

Solutions

$\sin\theta = \frac{-1}{2}$

The acute angle whose sine is $\frac{1}{2}$ is 30°. But sinθ is negative in the 3rd and 4th quadrants.

(a)



$\Rightarrow$ For sin θ = $\frac{-1}{2}$

θ = 210°

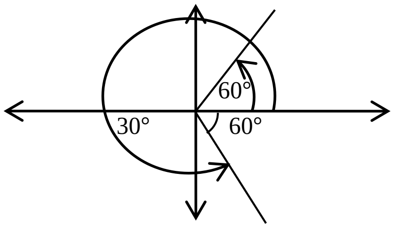
θ = 330°

(b) sec θ = 2

$\frac{1}{\cos\theta} = 2$

$\Rightarrow \frac{1}{2} = \cos\theta \quad \Rightarrow \cos\theta = \frac{1}{2}$

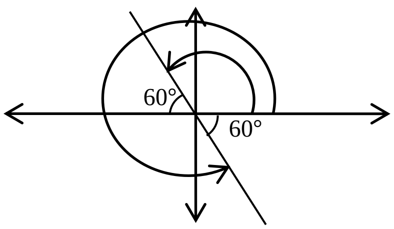
The acute angle whose cosine is $\frac{1}{2}$ is 60° but cos θ° is positive in the 1st and 4th quadrants.



For $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$

(c) $\tan\theta = -\sqrt{3}$

The acute angle whose tangent is $\sqrt{3}$ is 60° but $\tan\theta$ is negative in the 2^{nd} and 4^{th} quadrants.



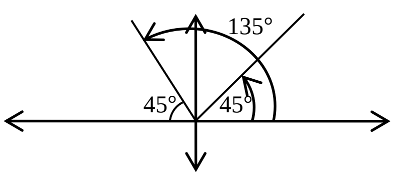
$\Rightarrow$ For $\tan\theta = -\sqrt{3}$, θ = 120°, 300°

(d) $\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

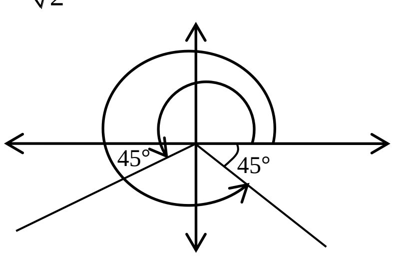
$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta\ \frac{-1}{\sqrt{2}}$

The acute angle whose sine is $\frac{1}{\sqrt{2}}$ is 45° but $\sin\theta$ is positive in the 1^{st} and 2^{nd} quadrants.



$\Rightarrow$ For $\sin\theta = \frac{1}{\sqrt{2}}$, θ = 45, 135

For $\sin\theta = \frac{-1}{\sqrt{2}}$



For sin θ = $\frac{-1}{\sqrt{2}}$

$\theta°$ = 225°, 315°

For $\sin^2\theta = \frac{1}{2}$, θ = 45, 135°, 225°, 315°

Example II

Solve the following equations for -180° ≤ θ ≤ 180°.

(a) Sin(2θ + 30) = 0.8

(b) $\tan^2$ θ + tan θ = 0

(c) $\sin^2$θ + sin θ = 0

(d) $2\sin^2$θ – sinθ – 1 = 0

Solution

(a) $\sin(2\theta + 30°) = 0.8$

$2\theta + 30° = \sin^{-1}(0.8)$

$2\theta + 30° = 53.1°, 126.9°$

$\Rightarrow$ 2θ = 23.1, 96.9

θ = 11.55, 48.45

For $\sin(2\theta + 30°) = 0.8$, θ = 11.55, 48.45.

(b) $\tan^2$θ + tan θ = 0

tan θ(tan θ + 1) = 0

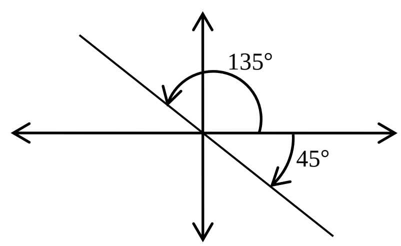
tan θ = 0 **OR** tan θ = -1

For tan θ = 0,

θ = $\tan^{-1}$0

θ = 0, -180, 180

For tan θ = -1, the acute angle whose tangent is 1 is 45°. But tan θ is negative in the 2^{nd} and 4^{th} quadrants.



For tan θ = -1, θ = 135°, -45°

$\Rightarrow$ $\tan^2\theta + \tan\theta = 0$

θ = -180°, -45°, 0, 135°, 180°

(c) **$\sin^2$ θ + sin θ = 0**

sin θ(sin θ +1) = 0

$\sin\theta = 0$, $\sin\theta = -1$

For sin θ = 0°, θ = 0, 180°, -180°

For $\sin\theta$ = -1,

The acute angle whose sine is 1 is 90°. Sine is negative in the 3^{rd} and 4^{th} quadrants.

For $\sin\theta = -1$, $\theta = -90$

For $\sin^2\theta + \sin\theta = 0°$, θ = -180°, -90°, 0°, 180°

(d) **$2\sin^2$ θ - sin θ – 1 = 0**

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2 \times 2}$$

$$\sin\theta = \frac{1 \pm 3}{4}$$

$\Rightarrow \sin\theta = 1, \ \sin\theta = \frac{-1}{2}$

For $\sin\theta = 1$,
$\theta = \sin^{-1}(1)$
$\theta = 90°$

For $\sin\theta = \frac{-1}{2}$,
θ = -30°, -150°
$\Rightarrow \theta$ = -30°, -150°, 90°

Example III
Solve the following equations from 0° to 360° inclusive.

(a) $\cos 3\theta = \frac{\sqrt{3}}{2}$

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

(c) $\sec 2\theta = 3$

(d) $4\cos 2\theta = 1$

(e) $\tan^2\theta = \frac{1}{3}$

(f) $\sin^2 2\theta = 1$

Solutions

(a) cos 3θ = $\frac{\sqrt{3}}{2}$

$3\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

3θ = 30°, 330°, 390°, 690°, 750°, 1050°
$\Rightarrow \theta$ = 10°, 110°, 130°, 230°, 250°, 350°

(b) $\tan(3\theta - 45°) = \frac{1}{2}$

$3\theta - 45 = \tan^{-1}\left(\frac{1}{2}\right)$

$3\theta - 45$ = 26.6, 206.6, 386.6, 566.6, 746.6, 926.6
$\Rightarrow \theta$ = 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°

(c) sec 2θ = 3

$$\frac{1}{\cos 2\theta} = 3$$

$$\frac{1}{3} = \cos 2\theta$$

$$2\theta = \cos^{-1}(\tfrac{1}{3})$$

2θ = 70.5°, 289.5°, 430.5°, 649.5°
θ = 35.25°, 144.75°, 215.25°, 324.75°

(d) tan²θ = $\frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\frac{1}{\sqrt{3}}$

For $\tan\theta = \frac{1}{\sqrt{3}}$, θ = 30°, 210°

For tan θ $-\frac{1}{\sqrt{3}}$, θ = 150°, 330°

$\Rightarrow$ When $\tan^2\theta = \frac{1}{3}$, θ = 30°, 150°, 210°, 230°

(e) sin²2θ = 1
$\sin 2\theta = \pm 1$
For $\sin 2\theta = 1$,
2θ = 90°, 450° $\Rightarrow$ θ = 45°, 225°
$\sin 2\theta$ = -1,
2θ = 270, 630 $\Rightarrow$ θ = 135°, 315°
$\Rightarrow$ When $\sin^2 2\theta = 1$,
θ = 45°, 135°, 225°, 315°

Example IV
Solve the following equations for values of θ from -180° to 180°

(a) $\tan\theta = \cot\theta + 3$

(b) $\sec\theta = 2\cos\theta$

(c) $5\sin\theta + 6\csc\theta = 17$

(d) $3\cos\theta + 2\sec\theta + 7 = 0$

Solution

(a) $\tan\theta = 4\cot\theta + 3$

$\tan\theta = \frac{4}{\tan\theta} + 3$

$\tan^2\theta = 4 + 3\tan\theta$.

$\tan^2\theta - 3\tan\theta - 4 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$\tan\theta = \frac{3 \pm 5}{2}$$

$\tan\theta = 4$, $\quad\tan\theta = -1$

When $\tan\theta = 4$,
$\theta = \tan^{-1}(4)$
θ = 76°, -104° (for -180° ≤ θ ≤ 180°)
When $\tan\theta = -1$,
$\theta = \tan^{-1}(-1)$ = -45°, 135° (for -180° ≤ θ ≤ 180°)
$\Rightarrow$ For $\tan\theta = 4\cot\theta + 3°$,
θ = -104°, -145°, 76°, 135°

(b) secθ = 2cosθ

$$\frac{1}{\cos\theta} = 2\cos\theta$$

$1 = 2\cos^2\theta$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\ \theta = \pm\frac{1}{\sqrt{2}}$$

For $\cos\theta = \frac{1}{\sqrt{2}}$, θ = 45°, -45°.

For $\cos\theta = \frac{-1}{\sqrt{2}}$, θ = 135°, -135°

∴ For sec θ = 2cos θ, θ = -135°, -45°, 45°, 135°.

(c) 5sin θ + 6cosec θ = 17

Solution

5sinθ + 6cosecθ = 17

$$5\sin\theta + \frac{6}{\sin\theta} = 17$$

$5\sin^2\theta + 6 = 17\sin\theta$

$5\sin^2\theta - 17\sin\theta + 6$

$$\sin\theta = \frac{17 \pm \sqrt{(-17)^2 - 4(5)\times 6}}{2\times 5}$$

$$\sin\theta = \frac{17 \pm \sqrt{289 - 120}}{10}$$

$$\sin\theta = \frac{17 \pm 13}{10}$$

sinθ = 3

sinθ = 0.4

$\theta = \sin^{-1}(0.4) \Rightarrow \theta$ = 23.6, 156.4

$\theta = \sin^{-1}(3) \Rightarrow \theta$ has no value since sinθ is maximum when it is 1

(d) 3cos θ +2sec θ +7 = 0

$$3\cos\ \theta + \frac{2}{\cos\theta} + 7 = 0$$

$3\cos^2\theta + 2 + 7\cos\theta = 0$

$3\cos^2\theta + 7\cos\theta + 2 = 0$

$$\cos\theta = \frac{-7 \pm \sqrt{(7)^2 - 4\times 3\times 2}}{2\times 3}$$

$$\cos\theta = \frac{-7 \pm 5}{6}$$

$$\cos\theta = \frac{-1}{3}$$

cosθ = -2

For cosθ = -2, θ has no values because the minimum of cosθ is -1

For $\cos\theta = \frac{-1}{3}$

θ = 109.5°, -109.5°.

Example IV

Solve the following equations from 0° to 360°

(a) $3 - \cos\theta = 2\sin^2\theta$

(b) $\cos^2\theta + \sin\theta + 1 = 0$

(c) $\sec^2\theta = 3\tan\theta - 1$

(d) $\text{cosec}^2\theta = 3 + \cot\theta$

(e) $3\tan^2\theta + 5 = 7\sec\theta$

Solutions

(a) $3 - \cos\theta = 2\sin^2\theta$

$3 - 3\cos\theta = 2(1 - \cos^2\theta)$

$3 - 3\cos\theta = 2 - 2\cos^2\theta$

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$$\cos\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times 2\times 1}}{2\times 2}$$

$$\cos\theta = \frac{3 \pm \sqrt{9-8}}{4}$$

cos θ = 1, **OR** $\cos\theta = \frac{1}{2}$

For cos θ = 1,

$\theta = \cos^{-1}(1)$

θ = 0°, 360°

For $\cos\theta = \frac{1}{2}$,

$\theta = \cos^{-1}(1/2)$

θ = 60°, 300°

$\Rightarrow$ For $3 - 3\cos\theta = 2\sin^2\theta$, θ = 0°, 60°, 300°, 360°

(b) $\cos^2\theta + \sin\theta + 1 = 0$

$1 - \sin^2\theta + \sin\theta + 1 = 0$

$\sin^2\theta - \sin\theta - 2 = 0$

$$\sin\ \theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times 1\times -2}}{2}$$

$$\sin\theta = \frac{1 \pm 3}{2}$$

sin θ = 2 **OR** sin θ = -1

For sin θ = 2, the value of θ is not defined because sin θ is maximum at 1

For sin θ= -1, θ = 270°

(c) sec$^2\theta$ = 3tan θ – 1

$\sec^2\theta = 1 + \tan^2\theta$

$\Rightarrow 1 + \tan^2\theta = 3\tan\theta - 1$

$\tan^2\theta - 3\tan\theta + 2 = 0$

$$\tan\theta = \frac{3 \pm \sqrt{(-3)^2 - 4\times1\times2}}{2\times1}$$

$$\tan\theta = \frac{3\pm1}{2}$$

$\tan\theta = 2$ **OR** $\tan\theta = 1$

For $\tan\theta = 2$,

$\theta = \tan^{-1}(2) = 63.4°, 243.4°$

For $\tan\theta = 1$,

$\theta = \tan^{-1}(1) = 45°, 225°$

∴ For $\sec^2\theta = 3\tan\theta - 1$, $\theta = 45°, 63.4°, 243.4°, 225°$.

(d) **$\text{cosec}^2\theta = 3 + \cot\theta$**

But $\text{cosec}^2\theta = 1 + \cot^2\theta$

$\Rightarrow 1 + \cot^2\theta = 3 + \cot\theta$

$\cot^2\theta - \cot\theta - 2 = 0$

$$\cot\theta = \frac{1 \pm \sqrt{(-1)^2 - 4\times1\times(-2)}}{2\times1}$$

$$\cot\theta = \frac{1\pm3}{2}$$

$\cot\theta = 2$ **OR** $\cot\theta = -1$

$\Rightarrow \tan\theta = \frac{1}{2}$ **OR** $\tan\theta = -1$

For $\tan\theta = \frac{1}{2}$, $\theta = \tan^{-1}(\frac{1}{2})$

$\theta = 26.6°, 206.6°$

For $\tan\theta = -1$, $\theta = 135°, 315°$

$\Rightarrow$ For $\text{cosec}^2\theta = 3 + \cot\theta$,

$\theta = 26.6°, 135°, 206.6°, 315°$

(e) $3\tan^2\theta + 5 = 7\sec\theta$

$3(\sec^2\theta - 1) + 5 = 7\sec\theta$

$3\sec^2\theta - 3 + 5 = 7\sec\theta$

$3\sec^2\theta - 7\sec\theta + 2 = 0$

$$\sec\theta = \frac{7 \pm \sqrt{(-7)^2 - 4\times3\times2}}{3\times2}$$

$$\sec\theta = \frac{7\pm5}{6}$$

$\sec\theta = 2$ **OR** $\sec\theta = \frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{2}$ **OR** $\cos\theta = 3$

For $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$

For $\cos\theta = 3$, θ is not defined because $\cos\theta$ is maximum at 1.

(f) $2\cot^2\theta + 8 = 7\text{cosec}\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\cot^2\theta = \text{cosec}^2\theta - 1$

$2(\text{cosec}^2\theta - 1) + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 2 + 8 = 7\text{cosec}\theta$

$2\text{cosec}^2\theta - 7\text{cosec}\theta + 6 = 0$

$$\text{cosec}\theta = \frac{7 \pm \sqrt{(-2)^2 - 4\times2\times2}}{2\times2}$$

$$\text{cosec}\theta = \frac{7\pm5}{4}$$

$\text{cosec}\theta = 3$, **OR** $\text{cosec}\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = \frac{1}{3}$, **OR** $\sin\theta = 2$

For $\sin\theta = \frac{1}{3}$, $\theta = 19.5, 160.5$

For $\sin\theta = 2$, $\theta = \sin^{-1}(2)$

The values of θ are not defined.

Example I (UNEB Questions)

Find all the values of θ, $0^0 \le \theta \le 360^0$, which satisfy the equation

$$\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0.$$

Solution

a) $\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

Dividing through by $\cos^2\theta$,

$\tan^2\theta - 2\tan\theta - 3 = 0$

$\tan^2\theta - 3\tan\theta + \tan\theta - 3 = 0$

$\tan\theta(\tan\theta - 3) + 1(\tan\theta - 3) = 0$

$(\tan\theta - 3)(\tan\theta + 1) = 0$

Either $\tan\theta - 3 = 0$

$\tan\theta = 3$

$\theta = \tan^{-1}(3)$

$\theta = 71.6^0, 251.6^0$

Or $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$

$\theta = 135^0, 315^0$

Example II (UNEB Question)

Solve $\cos\theta + \sin 2\theta = 0$ for $0^0 \le \theta \le 360^0$.

$\cos\theta + \sin 2\theta = 0$

$\cos\theta + 2\sin\theta\cos\theta = 0$

$\cos\theta(1 + 2\sin\theta) = 0$

Either $\cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\theta = 90^0, 270^0$

Or $1 + 2\sin\theta = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1}(-\frac{1}{2})$

$\theta = 210^0, 330^0$

For $0^0 \le \theta \le 360^0$, $\theta = 90^0, 210^0, 270^0, 330^0$

Example III (UNEB Question)

Solve $\cot^2\theta = 5(\text{cosec}\,\theta + 1)$ for $0° \le \theta \le 360°$

Solution

(a) $\cot^2\theta = 5(\text{cosec}\,\theta + 1)$

But $\cot^2\theta = \text{cosec}^2\theta - 1$

$\text{cosec}^2\theta - 1 = 5(\text{cosec}\,\theta + 1)$

$\text{cosec}^2\theta - 1 = 5\,\text{cosec}\,\theta + 5$

$\text{cosec}^2\theta - 5\,\text{cosec}\,\theta - 6 = 0$

$\text{cosec}^2\theta - 6\,\text{cosec}\,\theta + \text{cosec}\,\theta - 6 = 0$

$\text{cosec}\,\theta(\text{cosec}\,\theta - 6) + 1(\text{cosec}\,\theta - 6) = 0$

$\text{cosec}\,\theta - 6)(\text{cosec}\,\theta + 1) = 0$

Either $\text{cosec}\,\theta = 6$

$\frac{1}{\sin\theta} = 6$

$\sin\theta = \frac{1}{6}$

$\theta = 9.6°, 170.4°$

Or $\text{cosec}\,\theta + 1 = 0$

$\frac{1}{\sin\theta} = -1$

$\theta = 270°$

Hence $\theta = 9.6°$, $170.4°$ and $270°$

Example IV (UNEB Question)

Solve $2\sin 2x = 3\cos x$, for $-180^0 \le x \le 180°$.

Solution

$2\sin 2x = 3\cos x$

$2\sin 2x - 3\cos x = 0$

But $\sin 2x = 2\sin x\cos x$

$4\sin x\cos x - 3\cos x = 0$

$\cos x(4\sin x - 3) = 0$

$\cos x = 0$

$x = \cos^{-1}(0)$

$x = 90^0, -90$

$4\sin x - 3 = 0$

$\sin x = \frac{3}{4}$

$x = \sin^{-1}\left(\frac{3}{4}\right)$

$x = 48.6°, 131.4°$

$\Rightarrow$ $x = (^-90°, 48.6°, 90°, 131.4°)$ are the solutions to the equation $2\sin 2x = 3\cos x$

Example V (UNEB Question)

Solve the equation $\cos x + \cos 2x = 1$ for values of x from 0^0 to 360^0 inclusive

Solution

$\cos x + \cos 2x = 1$

But $\cos 2x = 2\cos^2 x - 1$

By substitution, we have

$\cos x + 2\cos^2 x - 1 = 1$

$2\cos^2 x + \cos x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\times 2\times(-2)}}{2\times 2}$$

$$\cos x = \frac{-1 \pm \sqrt{1^2 + 16}}{4}$$

$$= \frac{-1 \pm \sqrt{17}}{4}$$

Taking $\cos x = \frac{-1+\sqrt{17}}{4}$

$x = 38.7°, 321.3°$

Taking $\cos x = \frac{-1-\sqrt{17}}{4}$

$\cos x = -1.280776406$

(The values of x are not defined because x is maximum at 1)

Hence $x = 38.7^0, 321.3^0$

Example VI (UNEB Question)

Solve $7\tan\theta + \cot\theta = 5\sec\theta$ for $0° \le \theta \le 180°$.

Solution

(a) $7\tan\theta + \cot\theta = 5\sec\theta$

$$7\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\cos\theta}$$

Multiplying through by $\cos\theta\sin\theta$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + 1 - \sin^2\theta = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$6\sin^2\theta - 3\sin\theta - 2\sin\theta + 1 = 0$

$3\sin\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$

$(2\sin\theta - 1)(3\sin\theta - 1) = 0$

Either $2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\theta = 30^0, 150^0$

Or $3\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{3}$

$\theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\theta = 19.5^0, 160.5^0$

$\Rightarrow 19.5^0, 30^0, 150^0, 160.5^0$ are the solutions to the equation

Example VII (UNEB Question)

Solve the equation $4\cos x - 2\cos 2x = 3$ for $0^0 \leq x \leq \pi$.

Solution

$4\cos x - 2(2\cos^2 x - 1) = 3$

$4\cos x - 4\cos^2 x + 2 = 3$

$4\cos x - 4\cos^2 x - 1 = 0$

$4\cos^2 x - 4\cos x + 1 = 0$

$4\cos^2 x - 2\cos x - 2\cos x + 1 = 0$

$2\cos x(2\cos x - 1) - 1(2\cos x - 1) = 0$

$(2\cos x - 1)(2\cos x - 1) = 0$

$\Longrightarrow 2\cos x - 1 = 0$

$2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = 60°, 300°$

$x = \frac{\pi}{3}, \frac{3\pi}{3}.$

Elimination of θ from a set of equations

Example

Eliminate θ from the following equations:

(i) $x = a\cos\theta, y = b\sin\theta$

(ii) $x = a\cot\theta, y = b\sec\theta$

(iii) $x = a\tan\theta, y = b\tan\theta$

(iv) $x = 1 - \sin\theta, y = 1 + \cos\theta$

(v) $x = \sin\theta + \tan\theta, y = \tan\theta - \sin\theta$

(vi) $x\cos\theta + y\sin\theta = a, x\sin\theta - y\cos\theta = b$

Solution

(i) $x = a\cos\theta, y = b\sin\theta$

$\frac{x}{a} = \cos\theta, \ \frac{y}{b} = \sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) $x = a\cot\theta, y = b\operatorname{cosec}\theta$

$\frac{x}{a} = \cot\theta, = \frac{y}{b} = \operatorname{cosec}\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \left(\frac{x}{a}\right)^2 = \left(\frac{y}{b}\right)^2$

$1 + \frac{x^2}{a^2} = \frac{y^2}{b^2}$

(iii) $x = a\tan\theta, y = b\cos\theta$

$\frac{x}{a} = \tan\theta, \ \frac{y}{b} = \cos\theta \Rightarrow \frac{b}{y} = \sec\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \left(\frac{x}{a}\right)^2 = \frac{b^2}{y^2}$

$1 + \frac{x^2}{a^2} = \frac{b^2}{y^2}$

(iv) $x = 1 - \sin\theta, \quad y = 1 + \cos\theta$

$\sin\theta = 1 - x, \quad y - 1 = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$(1 - x)^2 + (y - 1)^2 = 1$

$\Rightarrow (x - 1)^2 + (y - 1)^2 = 1$

(v) $x = \sin\theta + \tan\theta$ ……………………. (i)

$y = \tan\theta - \sin\theta$ ………………..…… (ii)

Eqn (i) + Eqn (ii);

$\Rightarrow x + y = 2\tan\theta$

$\tan\theta = \frac{x+y}{2}$

Eqn (i) − Eqn (ii);

$x - y = 2\sin\theta$

$\frac{x-y}{2} = \sin\theta$

From $\tan\theta = \frac{x+y}{2}$

$\Rightarrow \cot\theta = \frac{2}{x+y}$

From $\sin\theta = \frac{x-y}{2}$

$\Rightarrow \operatorname{cosec}\theta = \frac{2}{x-y}$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \left(\frac{2}{x+y}\right)^2 = \left(\frac{2}{x-y}\right)^2$

$1 + \frac{4}{(x+y)^2} = \frac{4}{(x-y)^2}$

$\Rightarrow (x^2 - y^2)^2 = 16xy$

(vi) $x\cos\theta + y\sin\theta = a$ ..………………..(i)

$x\sin\theta - y\cos\theta = b$ ………………….. (ii)

From Eqn (i);

$\cos\theta = \dfrac{a - y\sin\theta}{x}$ …………….. (iii)

Substituting Eqn (iii) in Eqn (ii);

$$x\sin\theta - y\left(\frac{a - y\sin\theta}{x}\right) = b$$

$\Rightarrow x^2\sin\theta - ay + y^2\sin\theta = xb$

$(x^2 + y^2)\sin\theta = xb + ay$

$\sin\theta = \dfrac{bx + ay}{x^2 + y^2}$ …………………….. (iv)

Substitute Eqn (iv) in Eqn (iii)

$$\Rightarrow \cos\theta = \frac{a - y(\frac{bx+ay}{x^2+y^2})}{x}$$

$$\cos\theta = \frac{ax^2 + ay^2 - bxy - ay^2}{x(x^2 + y^2)}$$

$$\cos\theta = \frac{ax^2 - bxy}{x(x^2 + y^2)}$$

$$\cos\theta = \frac{ax - by}{x^2 + y^2}$$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{(bx + ay)^2}{(x^2 + y^2)^2} + \frac{(ax - by)^2}{(x^2 + y^2)^2} = 1$$

$(bx + ay)^2 + (ax - by)^2 = (x^2 + y^2)^2$

$b^2x^2 + 2abxy + a^2y^2 + a^2x^2 - 2abxy + b^2y^2$

$= (x^2 + y^2)^2$

$(a^2 + b^2)x^2 + (a^2 + b^2)y^2 = (x^2 + y^2)^2$

$(x^2 + y^2)(a^2 + b^2) = (x^2 + y^2)^2$

$a^2 + b^2 = x^2 + y^2$

Proving Trigonometric Identities

(i) $\sec\theta + \text{cosec}\theta\cot\theta = \sec\theta\,\text{cosec}^2\theta$

(ii) $\sin^2\theta(1 + \sec^2\theta) = \sec^2\theta - \cos^2\theta$

(iii) $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{1}{\cos ec\theta + \cot\theta}$

(iv) $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \text{cosec}\theta} = \dfrac{1}{\sin\theta + \cos\theta}$

(v) $\sec^2\theta = \dfrac{\text{cosec}\theta}{\text{cosec}\theta - \sin\theta}$

(vi) $\dfrac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

(vii) $\dfrac{1 + \sin\theta}{1 - \sin\theta} = (\sec\theta + \tan\theta)^2$

(viii) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha} = \cot\alpha\tan\beta$

Solution

(a) $\sec\theta + \text{cosec}\theta\cot\theta$

$$= \frac{1}{\cos\theta} + \frac{1}{\sin\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1}{\cos\theta} + \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\cos\theta} \times \frac{1}{\sin^2\theta}$$

$$= \sec\theta\,\text{cosec}\theta$$

(b) $\sin^2\theta(1 + \sec^2\theta)$

$$= \sin^2\theta + \sin^2\theta\sec^2\theta$$

$$= \sin^2\theta + \frac{\sin^2\theta}{\cos^2\theta}$$

$= \sin^2\theta + \tan^2\theta$

$= \sin^2\theta + \sec^2\theta - 1$

$= 1 - \cos^2\theta + \sec^2\theta - 1$

$= \sec^2\theta - \cos^2\theta$

(c) $\dfrac{1 - \cos\theta}{\sin\theta}$

$$= \frac{1 - \cos\theta}{\sin\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \frac{1 - \cos^2\theta}{\sin\theta + \sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta}{\sin\theta + \sin\theta\cos\theta}$$

$$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta}}$$

$$= \frac{1}{\text{cosec}\theta + \cot\theta}$$

(d) $\dfrac{\tan\theta + \cot\theta}{\sec\theta + \text{cosec}\theta}$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} + \frac{1}{\sin\theta}}$$

$$= \frac{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}{\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}} = \frac{1}{\sin\theta + \cos\theta}$$

(e) $\sec^2\theta = \dfrac{1}{\cos^2\theta}$

$$\sec^2\theta = \frac{1}{1-\sin^2\theta}$$

$$= \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}}$$

$$= \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta}$$

(f) $\dfrac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

$$\frac{1+\sin\theta}{\cos\theta} = \frac{(1+\sin\theta)\cos\theta}{\cos\theta\times\cos\theta}$$

$$= \frac{\cos\theta + \sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta\cos\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$

(g) $\dfrac{1+\sin\theta(1+\sin\theta)}{1-\sin\theta(1+\sin\theta)}$

$$= \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$$

$$= (\sec\theta + \tan\theta)^2$$

(h) $\dfrac{\cot\alpha + \tan\beta}{\cot\beta + \tan\alpha}$

$$= \frac{\frac{\cos\alpha}{\sin\alpha} + \frac{\sin\beta}{\cos\beta}}{\frac{\cos\beta}{\sin\beta} + \frac{\sin\alpha}{\cos\alpha}}$$

$$= \frac{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\sin\beta}}$$

$$= \frac{\cos\alpha\sin\beta}{\cos\beta\sin\alpha}$$

$$= \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\cos\beta}$$

$$= \cot\alpha\tan\beta$$

Formulae for sin(A ± B), cos(A ± B), and tan(A ± B)

sin(*A* + *B*) = sin*A*cos*B* + cos*A*sin*B*
sin(*A* – *B*) = sin*A*cos*B* – cos*A*sin*B*
cos(*A* + *B*) = cos*A*cos*B* – sin*A*sin*B*
cos(*A* – *B*) = cos*A*cos*B* + sin*A*sin*B*

tan(*A* + *B*) = $\dfrac{\tan A + \tan B}{1 - \tan A + \tan B}$

tan(*A* – *B*) = $\dfrac{\tan A - \tan B}{1 + \tan A + \tan B}$

Examples

Find the values of the following:

(a) cos(45° – 30°)
(b) cos 105°
(c) cos 75°
(d) sin(60° + 45°)
(e) sin 15°

Solution

(a) cos(45° – 35°)

= cos45° cos30° + sin45°sin30°

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{(1+\sqrt{3})2\sqrt{2}}{(2\sqrt{2})(2\sqrt{2})}$$

$$= \frac{2\sqrt{2}+2\sqrt{6}}{8} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

(b) sin(30° +45°)

= sin30 cos45 + cos30 sin 45

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{4} = \frac{\sqrt{6}+\sqrt{2}}{2}$$

(c) cos 105°

= cos(60° + 45°)

= cos60 cos45 – sin60 sin45

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad = \frac{(1-\sqrt{3})\sqrt{2}}{2\sqrt{2}\cdot\sqrt{2}}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

(d) cos 75°

= cos(30° + 45°)

= cos30° cos45° – sin30° sin45°

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)2\sqrt{2}}{2\sqrt{2}\times 2\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(f) sin(60° + 45°)

= sin60 cos45 + cos60 sin45

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3}+1)2\sqrt{2}}{2\sqrt{2}\times 2\sqrt{2}}$$

$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{8} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(f) sin 15°

= sin(45 – 30)

= sin45 cos30 – cos45 sin30

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{2\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3}-1)2\sqrt{2}}{8} \quad = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Example II

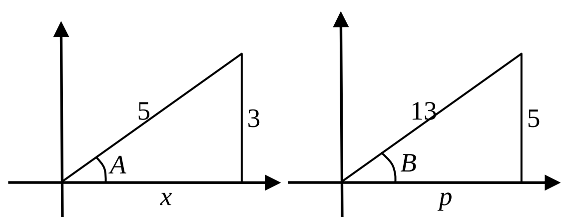
If sin $A = \frac{3}{5}$ and sin$B = \frac{5}{13}$, where A and B are acute angles, find the values of the following:

(a) sin(*A* + *B*)

(b) cos(*A* + *B*)

(c) cot(*A* + *B*)

Solution



$x^2 + 3^2 = 5^2$
$x^2 + 9 = 25$
$x^2 = 16$
$x = 4$

$p^2 + 5^2 = 13^2$
$p^2 + 25 = 169$
$p^2 = 144$
$p = 12$

$\Rightarrow$ $\sin A = \frac{3}{5}$; $\cos A = \frac{4}{5}$; $\tan A = \frac{3}{4}$

$\sin B = \frac{5}{13}$; $\cos B = \frac{12}{13}$; $\tan B = \frac{5}{12}$

sin(A + B) = sin$A$ cos$B$ + cos$A$ sin$B$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{56}{65}$$

(b) cos(*A* + *B*) = cos$A$ cos$B$ – sin$A$ sin$B$

$$= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13}$$

$$= \frac{48}{65} - \frac{15}{65}$$

$$= \frac{33}{65}$$

(c) cot(A + B) = $\frac{1}{\tan(A+B)}$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\Rightarrow \cot(A+B) = \frac{1-\tan A\tan B}{\tan A + \tan B}$$

$$= \frac{1-\frac{3}{4}\times\frac{5}{12}}{\frac{3}{4}+\frac{5}{12}}$$

$$= \frac{1-\frac{15}{48}}{7/6} = \frac{33/16}{7/6}$$

$$= \frac{33}{16}\times\frac{6}{7}$$

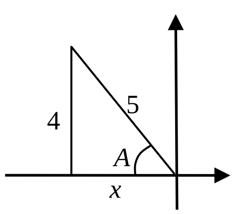
$$= \frac{99}{56}$$

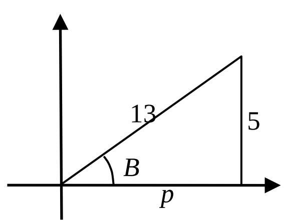
Example III

If sin$A$ = $\frac{4}{5}$, cos$B$ = $\frac{12}{13}$, where A is obtuse and B is acute, find the values of:

- **(a) sin(A – B)**
- **(b) tan(A – B)**
- **(c) tan(A + B)**

Solutions





$x^2 + 4^2 = 5^2$
$x^2 + 16 = 25$
$x^2 = 9$
$x = 3$

$p^2 + 12^2 = 13^2$
$p^2 + 144 = 169$
$p^2 = 25$
$p = 5$

A is obtuse

$\Rightarrow \sin A = \frac{4}{5}$; cos$A$ = $\frac{-3}{5}$; tan$A$ = $\frac{-4}{3}$

B is acute

$\Rightarrow$ sin$B$ = $\frac{5}{13}$; cos$B$ = $\frac{12}{13}$; tan$B$ = $\frac{5}{12}$

Sin(A – B) = sin$A$ cos$B$ – cos$A$ sin$B$

$$= \frac{4}{5}\times\frac{12}{13} - \frac{-3}{5}\times\frac{5}{13}$$

$$= \frac{48}{65} + \frac{15}{65}$$

$$= \frac{63}{65}$$

(b) tan(A – B) = $\frac{\tan A - \tan B}{1+\tan A\tan B}$

$$= \frac{\frac{-4}{3}-\frac{5}{12}}{1+\frac{-4}{3}\times\frac{5}{12}}$$

$$= \frac{-7/4}{4/9} = \frac{-63}{16}$$

(c) tan(A + B) = $\frac{\tan A + \tan B}{1-\tan A\tan B}$

$$= \frac{\frac{-4}{3}+\frac{5}{12}}{1-\frac{-4}{3}\times\frac{5}{12}}$$

$$= \frac{\frac{-11}{12}}{1+\frac{20}{36}}$$

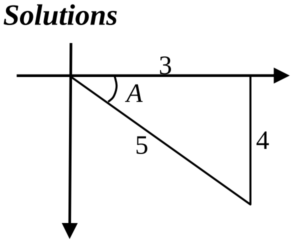
$$= \frac{\frac{-11}{12}}{\frac{56}{36}} = \frac{-33}{56}$$

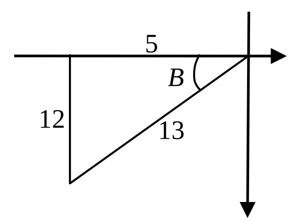
Example III

If cos$A$ = $\frac{3}{5}$ and tan$B$ = $\frac{12}{5}$; where A and B are reflex angles. Find the values of:

- **(a) sin(A – B)**
- **(b) tan(A – B)**
- **(c) cos(A + B)**

Solutions





A and B are reflex

$\Rightarrow \cos A = \frac{3}{5}$; $\sin A = \frac{-4}{5}$; $\tan A = \frac{-4}{3}$

$\cos B = \frac{-5}{13}$, $\sin B = \frac{-12}{13}$; $\tan B = \frac{12}{5}$

(a) sin(A – B) = sin A cos B – cos A sin B

$$= \frac{-4}{5}\times\frac{-5}{13} - \frac{3}{5}\times\frac{-12}{13}$$

$$= \frac{20}{65} + \frac{36}{65}$$

$$= \frac{56}{65}$$

(b) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{-4}{3} - \frac{12}{5}}{1 + \frac{-4}{3} \times \frac{12}{5}}$$

$$= \frac{-56/15}{-11/5} \qquad = \frac{56}{33}$$

(c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{3}{5} \times \frac{-5}{13} - \frac{-4}{5} \times \frac{-12}{13}$$

$$= \frac{-15}{65} - \frac{48}{65}$$

$$= \frac{-63}{65}$$

Example IV

From the following, find the values of tan x

(a) **$\sin(x + 45°) = 2\cos(x + 45°)$**

(b) **$2\sin(x - 45°) = \cos(x + 45°)$**

(c) **$\tan(x - A) = \frac{3}{2}$**, where $\tan A = 2$

(d) **$\sin(x + 30°) = \cos(x + 30°)$**

Solution

(a) **$\sin(x + 45°) = 2\cos(x + 45°)$**

$\sin x \cos 45° + \cos x \sin 45° = 2(\cos x \cos 45° - \sin x \sin 45°)$

$$\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = 2(\cos x \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sin x)$$

$$\frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x = 2(\tfrac{\sqrt{2}}{2} \cos x - \tfrac{\sqrt{2}}{2} \sin x)$$

$$\frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x = \sqrt{2} \cos x - \sqrt{2} \sin x$$

$$\left(\tfrac{\sqrt{2}}{2} + \sqrt{2}\right) \sin x = \sqrt{2} \cos x - \frac{\sqrt{2}}{2} \cos x$$

$$\frac{3\sqrt{2}}{2} \sin x = \frac{\sqrt{2}}{2} \cos x$$

$3\sin x = \cos x$

$$\frac{3\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$3 \tan x = 1$

$$\tan x = \frac{1}{3}$$

(b) $2\sin(x - 45°) = \cos(x + 45°)$

$2(\sin x \cos 45 - \cos x \sin 45) = \cos x \cos 45 - \sin x \sin 45$

$$2\left(\sin x(\tfrac{\sqrt{2}}{2}) - \cos x(\tfrac{\sqrt{2}}{2}) \;=\; \cos x(\tfrac{\sqrt{2}}{2}) - \sin x(\tfrac{\sqrt{2}}{2})\right)$$

$$\sqrt{2} \sin x - \sqrt{2} \cos x = \tfrac{\sqrt{2}}{2} \cos x - \tfrac{\sqrt{2}}{2} \sin x$$

$$\sqrt{2} \sin x + \tfrac{\sqrt{2}}{2} \sin x = \sqrt{2} \cos x + \tfrac{\sqrt{2}}{2} \cos x$$

$$\tfrac{3\sqrt{2}}{2} \sin x = \tfrac{3\sqrt{2}}{2} \cos x$$

$\tan x = 1$

(c) **$\tan(x - A) = \dfrac{3}{2}$, $\tan A = 2$**

$$\frac{\tan x - \tan A}{1 + \tan x \tan A} = \frac{3}{2}$$

$$\frac{\tan x - 2}{1 + 2\tan x} = \frac{3}{2}$$

$2(\tan x - 2) = 3(1 + 2\tan x)$

$2\tan x - 4 = 3 + 6\tan x$

$4\tan x = -7$

$$\tan x = \frac{-7}{4}$$

(d) **$\sin(x + 30) = \cos(x + 30)$**

$\sin x \cos 30 + \cos x \sin 30 = \cos x \cos 30 - \sin x \sin 30$

$$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$$

$$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \sin x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \cos x$$

$$\sin x \left(\frac{\sqrt{3}+1}{2}\right) = \cos x\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\frac{\sin x}{\cos x} = \frac{\frac{\sqrt{3}-1}{2}}{\frac{\sqrt{3}+1}{2}}$$

$$\tan x = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

$$\tan x = \frac{(\sqrt{3} - 1)(1 - \sqrt{3})}{(\sqrt{3} + 1)(1 - \sqrt{3})}$$

$$\tan x = \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{-2}$$

$\tan x = 2 - \sqrt{3}$

Example V

Solve the following equations for $0° \leq \theta \leq 360°$

(a) **$2\sin x = \cos(x + 60°)$**

(b) **$\cos(x + 45°) = \cos x$**

(c) **$\sin(x - 30°) = \dfrac{1}{2} \cos x$**

(d) **$3\sin(x + 10°) = 4\cos(x - 10°)$**

Solutions

(a) **$2\sin x = \cos(x + 60°)$**

$2\sin x = \cos x \cos 60° - \sin x \sin 60°$

$2\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$2\sin x + \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}\cos x$

$(4 + \sqrt{3})\sin x = \cos x$

$\tan x = \frac{1}{4+\sqrt{3}}$

$x = 9.9°, 189.9°$

(b) cos(x + 45°) = cos x

$\cos x \cos 45° - \sin x \sin 45° = \cos x$

$\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \cos x$

$\frac{\sqrt{2}}{2}\cos x + \cos x + \cos x = \frac{\sqrt{2}}{2}\sin x$

$(\frac{\sqrt{2}}{2} + 1)\cos x = \frac{\sqrt{2}}{2}\sin x$

$(\frac{\sqrt{2}+2}{2})\cos x = \frac{\sqrt{2}}{2}\sin x$

$\frac{\sqrt{2}+2}{\sqrt{2}} = \frac{\sin x}{\cos x}$

$\frac{\sqrt{2}+2}{\sqrt{2}} = \tan x$

$x = 67.5°, 247.5°$

(c) sin(x + 30) = $\frac{1}{2}$ cos x

$\sin x \cos 30 - \cos x \sin 30 = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2}\cos x$

$\frac{\sqrt{3}}{2}\sin x = \cos x$

$\frac{\sin x}{\cos x} = \frac{2}{\sqrt{3}}$

$\tan x = \frac{2}{\sqrt{3}}$

$x = 49.1°, 229.1°$

(d) 2sin(x + 10°) = 4cos(x – 10°)

$2(\sin x \cos 10 - \cos x \sin 10)$
$= 4(\cos x \cos 10° + \sin x \sin 10°)$

$2\sin x\cos 10 - 2\cos x \sin 10 = 4\cos x \cos 10 + 4\sin x \sin 10$

$2\sin x \cos 10 - 4\sin x \sin 10$
$= 4\cos x \cos 10 + 2\cos x \sin 10$

$\sin x(2\cos 10 - 4\sin 10) = \cos x(4\cos 10 + 2\sin 10)$

$\frac{\sin x}{\cos x} = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$

$\tan x = \frac{4\cos 10 + 2\sin 10}{2\cos 10 - 4\sin 10}$

$x = 73.4°, x = 253.4°$

Example VI

If tan(x + 45°) = 2, find the value of tan x

Solution

$\tan(x + 45°) = 2.$

$\frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$

$\frac{\tan x + 1}{1 - \tan x} = 2$

$\tan x + 1 = 2(1 - \tan x)$

$\tan x + 1 = 2 - 2\tan x$

$3\tan x = 1$

$\tan x = \frac{1}{3}$

Example VII

If tan(A + B) = $\frac{1}{7}$ and tan A = 3, find the value of tan B.

$\tan(A + B) = \frac{1}{7}$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{7}$

$\tan A = 3$

$\frac{3 + \tan B}{1 - 3\tan B} = \frac{1}{7}$

$7(3 + \tan B) = 1 - 3\tan B$

$21 + 7\tan B = 1 - 3\tan B$

$10\tan B = -20$

$\tan B = -2$

Example VIII

Express the following as single trigonometric ratios.

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

(b) $\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

(c) $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

(d) $\frac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

(e) $\frac{1}{2}\cos 75 + \frac{\sqrt{3}}{2}\sin 75$

(f) $\frac{1 - \tan 15}{1 + \tan 15}$

Solutions

(a) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

$= \cos 60 \cos x - \sin 60 \sin x$

$\cos(60 + x)$

$\Rightarrow \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \cos(60 + x)$

(b) $\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}$

$= \frac{\tan 60 + \tan x}{1 - \tan 60 \tan x}$

$= \tan(60 + x)$

(c) $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$

$= \cos 45 \sin x + \sin 45 \cos x$

$= \cos(45 – x)$

(d) $\frac{1}{\cos 24 \cos 15 - \sin 24 \sin 15}$

$= \frac{1}{\cos(24 + 15)}$

$= \frac{1}{\cos 39}$

$= \sec 39°$

(e) $\frac{1}{2}\cos 75 + \frac{\sqrt{3}}{2}\sin 75$

cos 60° cos 75° + sin 60° sin 75°

cos 75° cos 60° + sin 75° sin 60°

cos(75° – 60°)

cos 15°

(f) $\frac{1 - \tan 15}{1 + \tan 15} = \frac{\tan 45 - \tan 15}{1 + \tan 45 \tan 15}$

$= \tan(45 – 15)$

$= \tan(30)$

Example IX

Prove the following identities:

(i) sin(*A* + *B*) + sin(*A* – *B*) = 2sin *A* cos *B*

(ii) cos(A + B) – cos(A – B) = -2sin A sin B

(iii) tan *A* + tan *B* = $\frac{\sin(A + B)}{\cos A \cos B}$

(iv) tan(A + B + C) = $\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$

Hence prove that if *A*, *B*, and *C* are angles of a triangle, then tan *A* + tan *B* + tan *C* = tan *A* tan *B* tan *C*

Solution

sin (*A* + *B*) + sin(*A* – *B*)

sin *A* cos *B* + cos *A* sin *B* + sin *A* cos*B* – cos*A* sin*B*

= 2sin *A* cos *B*

$\Rightarrow$ sin(*A* + *B*) + sin(*A* – *B*) = 2sin *A* cos *B*

(ii) cos(*A* + *B*) – cos(*A* – *B*)

cos *A* cos *B* – sin *A* sin *B* – (cos *A* cos *B* + sin *A* sin *B*)

= -2sin A sin *B*

$\Rightarrow$ cos(A + B) – cos(A – B) = -2sin A sin B

(iii) tan A + tan B

$= \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$

$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$

$= \frac{\sin(A + B)}{\cos A \cos B}$

$\Rightarrow \tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B}$

(iv) tan(A + B + C)

Let *B* + *C* = D

$\tan(A + D) = \frac{\tan A + \tan D}{1 - \tan A \tan D}$

$= \frac{\tan A + \tan(B + C)}{1 - \tan A \tan(B + C)}$

$= \frac{\tan A + \frac{\tan B + \tan C}{1 - \tan B \tan C}}{1 - \tan A(\frac{\tan B + \tan C}{1 - \tan B \tan C})}$

$= \frac{\frac{\tan A - \tan A \tan B \tan C + \tan B + \tan C}{1 - \tan B \tan C}}{\frac{1 - \tan B \tan C - \tan A \tan B - \tan A + \tan C}{1 - \tan B \tan C}}$

$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$

Since *A*, *B*, and *C* are angles of a triangle, then

A + *B* + *C* = 180°

tan(*A* + *B* + *C*) = tan 180°

tan(*A* + *B* + *C*) = 0

$\Rightarrow \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B} = 0$

tan *A* + tan B + tan *C* – tan *A* tan *B* tan *C* = 0

tan *A* + tan B + tan *C* = tan *A* tan *B* tan *C*.

Example (UNEB Question)

Without using tables or calculator, evaluate tan 15°

Solution

tan 15° = tan(45° – 30°)

$= \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$

$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$$= \frac{(\sqrt{3}-1)(1-\sqrt{3})}{(\sqrt{3}+1)(1-\sqrt{3})}$$

$$= \frac{\sqrt{3}-3-1+\sqrt{3}}{1-3}$$

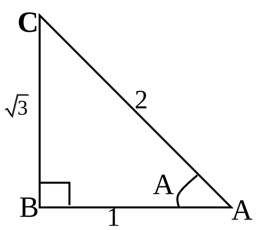
$$= \frac{2\sqrt{3}-4}{-2} = 2-\sqrt{3}$$

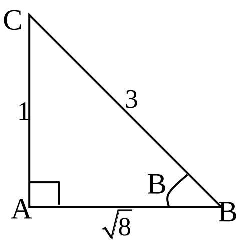
Example (UNEB Question)

The acute angles A and B are such that cos***A*** = ½, sin ***B*** = 1/3. Show without the use of tables or calculator, show that

$$\tan(A+B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Solution





$$\tan B = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8}$$

$$= \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$$

$$\tan B = \frac{\sqrt{3}}{2}$$

From compound angle formula,

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$= \frac{\sqrt{3}+\frac{\sqrt{2}}{4}}{1-(\sqrt{3}\times\frac{\sqrt{2}}{4})}$$

$$= \frac{\frac{4\sqrt{3}+\sqrt{2}}{4}}{\frac{4-\sqrt{3}\times\sqrt{2}}{4}}$$

$$= \frac{4\sqrt{3}+\sqrt{2}}{4} \times \frac{4}{4-\sqrt{6}}$$

$$= \frac{\left(4\sqrt{3}+\sqrt{2}\right)}{\left(4-\sqrt{6}\right)}$$

$$= \frac{\left(4\sqrt{3}+\sqrt{2}\right)\left(4+\sqrt{6}\right)}{\left(4-\sqrt{6}\right)\left(4+\sqrt{6}\right)}$$

$$= \frac{16\sqrt{3}+4\sqrt{18}+4\sqrt{2}+\sqrt{12}}{16-6}$$

$$= \frac{16\sqrt{3}+4\times3\sqrt{2}+4\sqrt{2}+2\sqrt{3}}{10}$$

$$= \frac{18\sqrt{3}+16\sqrt{2}}{10}$$

$$= \frac{9\sqrt{3}+8\sqrt{2}}{5}$$

Double angle & Triple angle formulae

By writing $A = B$ in the additional formulae for sine, cosine, and tangent, we obtain the double angle formula for each of them.

$$\sin(A + B) = \sin A \cos B + \cos A \cos B$$

$\Rightarrow \sin 2A = \sin(A + A)$
$= \sin A \cos A + \cos A \sin A$
$= 2\sin A \cos A$

sin 2*A* = 2sin *A* cos *A*

$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cos(A + A) = \cos A \cos A - \sin A \sin A.$
$= \cos^2 A - \sin^2 A$
But $\cos^2 A = 1 - \sin^2 A$
$\Rightarrow \cos 2A = 1 - \sin^2 A - \sin^2 A$
$= 1 - 2\sin^2 A$
But when $\sin^2 A = 1 - \cos^2 A$
$\cos^2 A = \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$

cos ²*A* = 2cos²A – 1 OR cos ²*A* = 1 – 2sin²A

$$\tan(A + B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\text{; where } A = B$$

$$\tan(A + A) = \frac{\tan A + \tan A}{1-\tan A \tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

sin 3*A* = sin(*A* + 2*A*)
$= \sin A \cos 2A + \cos A \sin 2A$
$= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A \cos A)$
$= \sin A - 2\sin^3 A + 2\cos^2 A \sin A$
$= \sin A - 2\sin^3 A + 2(1 - \sin^2 A)\sin A$
$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$
$= 3\sin A - 4\sin^3 A$

$\boxed{\sin 3A = 3\sin A - 4\sin^3 A}$

cos 3*A* = cos(2*A* + *A*)

$= \cos 2A \cos A - \sin 2A \sin A$
$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$
$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$
$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$
$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$
$= 4\cos^3 A - 3\cos A$

$\Rightarrow$ **cos 3*A* = 4cos³*A* – 3cos *A***

tan 3*A* = tan(*A* + 2*A*)

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\tan A + \frac{2\tan A}{1-\tan^2 A}}{1 - \tan A\left(\frac{2\tan A}{1-\tan^2 A}\right)}$$

$$= \frac{\frac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\frac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\boxed{\Rightarrow \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$$

Example I

Simplify the following expressions

(i) **2sin 17 cos 17**

(ii) $\frac{2\tan 30°}{1-\tan^2 30°}$

(iii) **2cos²42 – 1**

(iv) $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

(v) $1 - 2\sin^2 22\frac{1}{2}°$

(vi) $\frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{\theta}{2}}$

(vii) **1 – 2sin²3θ**

(viii) $\frac{1-\tan^2 20}{\tan 20}$

(ix) **sec θ cosec θ**

(x) **2sin 2*A* cos 2*A***

Solutions

(i) **sin 2(17°)** = 2sin 17° cos 17°

sin 34° = 2sin 17° cos 17°

$\Rightarrow$ 2sin 17 cos 17 = sin 34

(ii) **tan(30° + 30°) =** $\frac{\tan 30° + \tan 30°}{1 - \tan 30° \tan 30°}$

$\tan 60° = \frac{2\tan 30°}{1-\tan^2 30°}$

$\frac{2\tan 30°}{1-\tan^2 30°} = \tan 60°$

(iii) **2cos²42° – 1**

$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2(42°) = 2\cos^2 42° - 1$
$\cos 84° = 2\cos^2 42° - 1$
$2\cos^2 42° - 1 = \cos 84°$

(iv) $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin 2(\frac{1}{2}\theta) = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$
$\sin\theta = 2\sin\frac{1}{2}\theta\ \cos\frac{1}{2}\theta$
$\Rightarrow \sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$
$2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = \sin\theta$

(v) **1 – 2sin²22½°**

$\cos 2A = 1 - 2\sin^2 A$
$\cos 2(22½) = 1 - 2\sin^2 22½$
$\cos 45 = 1 - 2\sin^2 22½$
$1 - 2\sin^2 22½ = \cos 45$

(vi) $\frac{2\tan\frac{1}{2}\theta}{1-2\tan^2\frac{1}{2}\theta} = \frac{\tan\frac{1}{2}\theta + \tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta}$

$= \tan(\frac{1}{2}\theta + \frac{1}{2}\theta +)$

$= \tan\theta$

(vii) $1 - 2\sin^2\theta$

$\cos 2(3\theta) = 1 - 2\sin^2 3\theta$
$\cos 6\theta = 1 - 2\sin^2 3\theta$
$1 - 2\sin^2 3\theta = \cos 6\theta$

(viii) $\frac{1-\tan^2 20}{\tan 20}$

$\tan 40 = \tan(20 + 20)$

$= \frac{2\tan 20}{1-\tan^2 20}$

$\frac{1}{\tan 40} = \frac{1-\tan^2 20}{2\tan 20}$

$\frac{2}{\tan 40} = \frac{1-\tan^2 20}{\tan 20}$

$$2\cot 40 = \frac{1-\tan^2 20}{\tan 20}$$

$$\frac{1-\tan^2 20}{\tan 20} = 2\cot 40$$

(ix) $\sec\theta \operatorname{cosec}\theta = \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$

$$= \frac{1}{\cos\theta \sin\theta}$$

But sin 2θ = $2\sin\theta\cos\theta$

$\frac{1}{2}$ sin 2θ = sin θ cos θ

$$\sec\theta \operatorname{cosec}\theta = \frac{1}{\frac{1}{2}\sin 2\theta}$$

$$\sec\theta \operatorname{cosec}\theta = \frac{2}{\sin 2\theta}$$

$$\sec\theta \operatorname{cosec}\theta = 2\operatorname{cosec} 2\theta$$

(x) 2sin 2*A* cos 2*A*

sin 4*A* = sin2(2*A*)

= 2sin 2*A* cos2*A*

$\Rightarrow$ 2sin 2*A* cos 2*A* = sin 4*A*

Example II

Evaluate the following without using tables or calculator:

(a) 2sin 15° cos 15°

(b) $2\cos^2 75° - 1$

(c) $\cos^2 22½° - \sin^2 22 ½°$

(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65}$

(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}}$

(f) $1-2\sin^2 67\frac{1}{2}$

Solution

(a) 2sin 15° cos 15° = sin 2(15°)

= sin 30°

$= \frac{1}{2}$

(b) $2\cos^2 75° - 1$ = cos 150°

= -cos 30°

$= \frac{-\sqrt{3}}{2}$

(c) $\cos^2 22½° - \sin^2 22 ½°$

= cos($22\frac{1}{2}$° + $22\frac{1}{2}$°)

= cos 45

$= \frac{\sqrt{2}}{2}$

(d) $\frac{1-2\cos^2 25}{1-2\sin^2 65} = \frac{-1(2\cos^2 25-1)}{1-2\sin^2 65}$

$$= \frac{-1(\cos 50°)}{\cos 130°}$$

$$= \frac{-1(\cos 50°)}{-\cos 50°} = 1$$

(e) $\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan(22\tfrac{1}{2}° + 22\tfrac{1}{2}°)$

$\tan 45° = 1$

$$\frac{2\tan 22\frac{1}{2}}{1-\tan^2 22\frac{1}{2}} = \tan 45° = 1$$

(f) $1-2\sin^2 67\frac{1}{2}$ = cos 135°

= -cos 45°

$= \frac{-\sqrt{2}}{2}$

Example III

Solve the following equations from $0 \le \theta \le 360°$

(a) cos 2θ + cos *θ* + 1 = 0

(b) sin 2*θ* cos *θ* + $\sin^2\theta$ = 1

(c) 2sin θ(5cos 2*θ* + 1) = 3 sin 2*θ*

(d) 3cot 2θ + cot θ = 1

(e) 4tan *θ* tan 2*θ* = 1

Solution

(a) cos 2θ + cos *θ* + 1 = 0

$$2\cos^2\theta - 1 + \cos\theta + 1 = 0$$

$$2\cos^2\theta + \cos\theta = 0$$

$$\cos\theta(2\cos\theta + 1) = 0$$

$$\cos\theta = 0,\quad \cos\theta = \frac{-1}{2}$$

For $\cos\theta = 0,\ \theta = 90°, 270°$

For $\cos\theta = \frac{-1}{2}$, *θ* = 120°, 240°

$\Rightarrow$ The solutions to the equation cos 2θ + cos *θ* + 1 = 0 are 90°, 120°, 240° and 270°.

(b) sin 2*θ* cos *θ* + $\sin^2\theta$ = 1

$$(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta = 1$$

$$2\cos^2\theta\sin\theta + \sin^2\theta = 1$$

$$2(1-\sin^2\theta)\sin\theta + \sin^2\theta = 1$$

$$2\sin\theta - 2\sin^3\theta + \sin^2\theta = 1$$

$$2\sin^3\theta - \sin^2\theta - 2\sin\theta + 1 = 0$$
$$\sin\theta = 1, \quad \sin\theta = -1$$
$$\sin\theta = \frac{1}{2}$$

For $\sin\theta = 1$, $\theta = 90°$
For $\sin\theta = -1$, $\theta = 270°$
For $\sin\theta = \frac{1}{2}$, θ = 30°, 150°

$\Rightarrow$ 30°, 90°, 150°, 270° are the solutions to the equation sin 2θ cos $\boldsymbol{\theta}$ + sin$^2\boldsymbol{\theta}$ = 1

(c) 2sin θ(5cos 2θ + 1) = 3 sin 2θ

$$2\sin\theta[5(2\cos^2\theta - 1) + 1] = 3\cdot 2\sin\theta\cos\theta$$
$$2\sin\theta(10\cos^2\theta - 5 + 1) = 6\sin\theta\cos\theta$$
$$20\cos^2\theta\sin\theta - 8\sin\theta = 6\sin\theta\cos\theta$$
$$20\cos^2\theta\sin\theta - 8\sin\theta - 6\sin\theta\cos\theta = 0$$
$$2\sin\theta[10\cos^2\theta - 3\cos\theta - 4] = 0$$
$$\sin\theta = 0, \quad \cos\theta = 0.8, \quad \cos\theta = \frac{-1}{2}$$

For sin θ = 0, θ = 0°, 180°, 360°
For cos $\theta = \frac{-1}{2}$, θ = 120°, 240°
For cos θ = 0.8, θ = 36.9°, 323.1°

$\Rightarrow$ 0, 36.9, 120, 180, 240, 323.1, 360 are the solutions to the equation 2sin θ(5cos 2θ + 1) = 3 sin 2θ

(d) 3cot 2$\boldsymbol{\theta}$ + cot $\boldsymbol{\theta}$ = 1

$$\frac{3}{\tan 2\theta} + \frac{1}{\tan\theta} = 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\Rightarrow 3\left(\frac{1 - \tan^2\theta}{2\tan\theta}\right) + \frac{1}{\tan\theta} = 1$$
$$3 - 3\tan^2\theta + 2 = 2\tan\theta$$
$$3\tan^2\theta + 2\tan\theta - 5 = 0$$
$$\tan\theta = 1, \quad \tan\theta = \frac{-5}{3}$$

For tanθ = 1, θ = 45°, 225°
For tan$\theta = \frac{-5}{3}$, θ = 121°, 301°.

(e) 4tan θ tan 2$\boldsymbol{\theta}$ = 1

$$4\tan\theta\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) = 1$$
$$\frac{8\tan^2\theta}{1 - \tan^2\theta} = 1$$
$$8\tan^2\theta = 1 - \tan^2\theta$$
$$9\tan^2\theta = 1$$
$$\tan\theta = \pm\frac{1}{3}$$

When $\tan\theta = \frac{1}{3}$, θ = 18.4°, 198.4°
When $\tan\theta = \frac{-1}{3}$, θ = 161.6°, 341.6°

t-formula

If *t* = $\tan\frac{x}{2}$**,**

$$\sin x = \frac{2t}{1 + t^2}, \qquad \cos x = \frac{1 - t^2}{1 + t^2}$$

And if *t* = tan *x*

$$\sin 2x = \frac{2t}{1 + t^2}, \qquad \cos 2x = \frac{1 - t^2}{1 + t^2}$$

Proof

If $t = \tan\frac{x}{2}$,

$$\sin x = \sin\left(\tfrac{x}{2} + \tfrac{x}{2}\right)$$
$$= \sin\tfrac{x}{2}\cos\tfrac{x}{2} + \cos\tfrac{x}{2}\sin\tfrac{x}{2}$$
$$= 2\sin\tfrac{x}{2}\cos\tfrac{x}{2}$$
$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}$$
$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}$$

Dividing through by $\cos^2\frac{x}{2}$

$$\sin x = \frac{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}$$
$$\sin x = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2} + 1}$$
$$= \frac{2t}{1 + t^2}$$
$$\boxed{\sin x = \frac{2t}{1 + t^2}}$$

$\cos x = \cos(\frac{x}{2} + \frac{x}{2})$

$= \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$

$= \cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{1}$

$\cos x = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}$

Dividing through by $\cos^2 \frac{x}{2}$

$$\cos x = \frac{\frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} + \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}$$

$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 1}$

$\cos x = \frac{1 - t^2}{1 + t^2}$

For $t = \tan x$

$\sin 2x = \frac{2\sin x \cos x}{1}$

$= \frac{2\sin x \cos x}{\sin^2 x + \cos^2 x}$

Dividing through by $\cos^2 x$

$\sin 2x = \frac{\frac{2\sin x \cos x}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}}$

$\sin 2x = \frac{2\tan x}{\tan^2 x + 1}$

$\sin 2x = \frac{2t}{1 + t^2}$

$\cos 2x = 2\cos^2 x - 1$

$= \frac{2}{\sec^2 x} - 1$

$= \frac{2 - \sec^2 x}{\sec^2 x}$

$= \frac{2 - (1 + \tan^2 x)}{\sec^2 x}$

$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

$\cos 2x = \frac{1 - t^2}{1 + t^2}$

Note: The t-formula is used to solve equations of the form $a\cos\theta + b\sin\theta = c$

Example I

Solve the following equations for $0 \le \theta \le 360°$

(a) $2\cos\theta + 3\sin\theta - 2 = 0$

(b) $3\cos\theta - 4\sin\theta + 1 = 0$

(c) $3\cos\theta + 4\sin\theta = 2$

(d) $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

Solution

(a) $2\cos\theta + 3\sin\theta = 2$

$\cos\theta = \frac{1 - t^2}{1 + t^2}$, $\sin\theta = \frac{2t}{1 + t^2}$, for $t = \tan\frac{\theta}{2}$

$2\left(\frac{1 - t^2}{1 + t^2}\right) + 3\left(\frac{2t}{1 + t^2}\right) = 2$

$2(1 - t^2) + 3(2t) = 2(1 + t^2)$

$2 - 2t^2 + 6t = 2 + 2t^2$

$4t^2 - 6t = 0$

$2t(2t - 3) = 0$

$t = 0$

$\Rightarrow \tan\frac{\theta}{2} = 0$ and $\tan\frac{\theta}{2} = \frac{3}{2}$

For $\tan\frac{\theta}{2} = 0$, $\frac{\theta}{2} = \tan^{-1}(0)$

$\frac{\theta}{2} = 0°, 180°, \ldots$

$\theta = 0, 360.$

For $\tan\frac{\theta}{2} = \frac{3}{2}$, $\frac{\theta}{2} = 56.3°$

$\theta = 112.6°$

$\Rightarrow$ 0°, 112.6°, and 360° are solutions to the equation $2\cos\theta + 3\sin\theta - 2 = 0$

(b) $3\cos\theta - 4\sin\theta + 1 = 0$

$3\left(\frac{1 - t^2}{1 + t^2}\right) - 4\left(\frac{2t}{1 + t^2}\right) + 1 = 0$

$3 - 3t^2 - 8t + 1 + t^2 = 0$

$-2t^2 - 8t + 4 = 0$

$t^2 + 4t - 2 = 0$

$t = \frac{-4 \pm \sqrt{(4)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$

$t = \frac{-4 \pm \sqrt{16 + 8}}{2}$

$t = -2 \pm \sqrt{6}$

$\tan\frac{\theta}{2} = -2 - \sqrt{6}$

$\tan\frac{\theta}{2} = -2 + \sqrt{6}$

For $t = -2 - \sqrt{6}$, $\tan\frac{\theta}{2} = -2 - \sqrt{6}$

$\frac{\theta}{2} = 102.7, 282.7$

$\theta = 205.4°$

When $t = \tan\frac{\theta}{2} = -2+\sqrt{6}$

$$\frac{\theta}{2} = \tan^{-1}(-2 + \sqrt{6})$$

$$\frac{\theta}{2} = 24.2°$$

$\theta = 48.4°$

$\Rightarrow \theta = 48.4°$ and 205.4° are the solutions to the equation

(c) $3\cos\theta + 4\sin\theta = 2$

$$3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)=2$$

$3 - 3t^2 + 8t = 2(1 + t^2)$

$3 - 3t^2 + 8t = 2 + 2t^2$

$5t^2 - 8t - 1 = 0$

$$t=\frac{8\pm\sqrt{(-8)^2-4\times5\times(-1)}}{2\times5}$$

$$t=\frac{8\pm\sqrt{64+20}}{10}$$

$$t=\frac{8\pm\sqrt{84}}{10}$$

t = -0.11652

t = 1.71652

For t = -0.11652, $\tan\frac{\theta}{2}$ = -0.11652

$$\frac{\theta}{2} = 173.4 \quad \Rightarrow \theta = 346.7°$$

$$\tan\frac{\theta}{2} = 1.71652$$

$$\frac{\theta}{2} = 59.8° \quad \Rightarrow \theta = 119.6°$$

$\Rightarrow$ 119.6° and 346.7° are solutions to the above equation.

(d) $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

$2 \times 2\sin\theta\cos\theta + 15\cos 2\theta = 10$

$2\sin 2\theta + 15\cos 2\theta = 10$

$2\sin 2\theta + 15\cos 2\theta = 0$

Let $t = \tan\theta$

$$\sin\theta=\frac{2t}{1+t^2} \text{ and } \cos\theta=\frac{1-t^2}{1+t^2}$$

$$2\left(\frac{2t}{1+t^2}\right)+15\left(\frac{1-t^2}{1+t^2}\right)=10$$

$4t + 15 - 15t^2 = 10 + 10t^2$

$25t^2 - 4t - 5 = 0$

$$t=\frac{4\pm\sqrt{(-4)^2-4\times25\times(-5)}}{2\times25}$$

t = 0.5343

t = -0.3743

For t = 0.5343

$\tan\theta$ = 0.5343

θ = 28.1°

θ = 208.1°

For t = -0.3743, $\tan\theta$ = -0.3743

$\theta = \tan^{-1}(0.3743)$

θ = 159.5°, 200.5°

$\Rightarrow$ 28.1°, 208.1°, 159.5° and 200.5° are the solutions to the above equation

The *R*- Formula

The *R*-formula is used to solve equations of the form $a\cos\theta + b\sin\theta = c$.

Rcos(θ ± α) = c
Rsin(θ ± α) = c

Where $R = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$

Example I

Solve the equation $3\cos\theta + 4\sin\theta = 2$ for 0 ≤ θ ≤ 360°

Solution

$R\cos(\theta-\alpha)=2$

$R(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=2$

$R\cos\theta\cos\alpha+R\sin\theta\sin\alpha=2$

By comparison

$R\cos\theta\cos\alpha=3\cos\theta$

$R\sin\theta\sin\alpha=4\sin\theta$

$\Rightarrow R\cos\alpha=3$ ……………………….. (i)

$R\sin\alpha=4$ ……………………….. (ii)

Eqn (ii) ÷ Eqn (1);

$$\Rightarrow \tan\alpha=\frac{4}{3}$$

$$\alpha=\tan^{-1}\left(\frac{4}{3}\right)$$

$\alpha = 53.1°$

$R^2\cos^2\alpha+R^2\sin^2\alpha=3^2+4^2$

$R^2(\cos^2\alpha+\sin^2\alpha)=25$

$R=5$

$R\cos(\theta-\alpha)=2$

$5\cos(\theta-53.1)=2$

$$\theta-53.1=\cos^{-1}\left(\frac{2}{5}\right)$$

θ – 53.1° = 66.4°, 293.6°

θ – 119.5°, 346.7°

Alternatively

$3\cos\theta + 4\sin\theta = 2$

$R\cos(\theta - \alpha) = 2$

$R = \sqrt{a^2 + b^2}$

$= \sqrt{(3)^2 + 4^2}$

$= 5$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$

$\alpha = 53.1$

$5\cos(\theta - 53.1) = 2$

$\cos(\theta - 53.1°) = \frac{2}{5}$

$\theta - 53.1° = 66.4°, 293.6°$

$\theta = 119.5°, 346.7°$

Example II

$\sin\theta + \sqrt{3}\cos\theta = 1$ for $0 \leq \theta \leq 360$

Solution

$R\sin(\theta + \alpha) = 1$

$R = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

$R\sin(\theta + \alpha) = 1$

$2\sin(\theta + 60°) = 1$

$\sin(\theta + 60°) = \frac{1}{2}$

$\theta + 60° = \sin(½)$

$\theta + 60° = 30, 150°$

$\theta = -30, 90°$

$\Rightarrow \theta = 90°$, and $330°$.

Example III

$\cos\theta - 7\sin\theta = 2$ for $0° \leq \theta \leq 360°$

Solution

$\cos\theta - 7\sin\theta = 2$

$R\cos(\theta + \alpha) = 2$

$R = \sqrt{1^2 + (-7)^2} = \sqrt{50}$

$\alpha = \tan^{-1}\left(\frac{7}{1}\right) \qquad \Rightarrow \alpha = 81.9°$

$\sqrt{50}\cos(\theta + 81.9°) = 2$

$\cos(\theta + 81.9°) = \frac{2}{\sqrt{50}}$

$\theta + 81.9° = 73.6°, 286.4°$

$\theta = -8.3°, 204.5°$

$\Rightarrow \theta = 204.5°, 351.7°$

Example IV

Solve: $5\sin\theta - 12\cos\theta = 6$

Solution

$R\sin(\theta - \alpha) = 6$

$R = \sqrt{5^2 + 12^2} = 13$

$\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

$\alpha = 67.4°$

$13\sin(\theta - 67.4) = 6$

$\sin(\theta - 67.4) = \frac{6}{13}$

$\theta - 67.4° = 27.5°, 152.5°$

$\theta = 94.9°, 219.9°$

Example V

Solve $\cos\theta + \sin\theta = \sec\theta$ for $0 \leq \theta \leq 360°$

Solution

$\cos\theta + \sin\theta = \frac{1}{\cos\theta}$

$\cos^2\theta + \sin\theta\cos\theta = 1$.............................(i)

But $\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Subsituting for $\cos^2\theta$ and $\sin\theta\cos\theta$ in Eqn (i);

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}\sin 2\theta = 1$

$\frac{1}{2}\cos 2\theta + \frac{1}{2}\sin 2\theta = \frac{1}{2}$

$\cos 2\theta + \sin 2\theta = 1$

$R\cos(2\theta - \alpha) = 1$

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\alpha = \tan^{-1}\left(\frac{1}{1}\right) = 45°$

$\sqrt{2}\cos(2\theta - 45°) = 1$

$\cos(2\theta - 45°) = \frac{1}{\sqrt{2}}$

$2\theta - 45° = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$2\theta - 45° = 45°, 315°, 405°$

$\theta = 45°, 180°, 225°$

Example VI

Solve the equation $4\cos\theta\sin\theta + 15\cos 2\theta = 10$

Solution

$4\cos\theta\sin\theta + 15\cos 2\theta = 10$

$2(2\sin\theta\cos\theta) + 15\cos2\theta = 10$
$2\sin2\theta + 15\cos2\theta = 10$
$R\sin(2\theta + \alpha) = 10$
$R = \sqrt{2^2 + 15^2}$
$= \sqrt{229}$
$\sqrt{229}\sin(2\theta + \alpha) = 10$
$\alpha = \tan^{-1}\left(\frac{15}{2}\right) = 82.4°$
$\sqrt{229}\sin(2\theta + 82.4°) = 10$
$\sin(2\theta + 82.4°) = \frac{10}{\sqrt{229}}$
$2\theta + 82.4° = \sin^{-1}\left(\frac{10}{\sqrt{229}}\right)$
$2\theta + 82.4° = 41.4°, 138.6°, 401.4°, 498.4°$
$\theta = 339.5°, 28.1°, 159.5°, 208°$

Example VII
Show that $3\cos\theta + 2\sin\theta$ can be written as $\sqrt{13}\cos(\theta - \alpha)$. Hence find the minimum and maximum values of the function, giving the corresponding values of θ from -180° to 180°

Solution
$3\cos\theta + 2\sin\theta$
$R\cos(\theta - \alpha)$
$R = \sqrt{a^2 + b^2}$
$R = \sqrt{3^2 + 2^2} = \sqrt{13}$
$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.7°$
$\Rightarrow 3\cos\theta + 2\sin\theta = R\cos(\theta - \alpha)$
$= \sqrt{13}\cos(\theta - 33.7)$
Let $y = \sqrt{13}\cos(\theta - 33.7)$
For the maximum value of y, $\cos(\theta - 33.7) = 1$
$\Rightarrow y_{max} = \sqrt{13}$
And for minimum value of y, $\cos(\theta - 33.7) = -1$
$\Rightarrow y_{min} = -\sqrt{13}$
For y_{max} $\cos(\theta - 33.7°) = 1$,
$\Rightarrow \theta - 33.7° = \cos^{-1}(1)$
$\theta - 33.7° = 0, 360°.$
$\theta = 33.7°$
For y_{min} $\cos(\theta - 33.7°) = -1$,
$\theta - 33.7° = 180°.$
$\theta = 213.7°$

Example VII
Find the maximum and minimum values of the following expressions, stating the value of θ for which they occur (from 0° to 360°)

(a) $8\cos\theta - 15\sin\theta$
(b) $4\sin\theta - 3\cos\theta$
(c) $\sin\theta - 6\cos\theta$
(d) $\cos(\theta + 60) - \cos\theta$

Solution
(a) $8\cos\theta - 15\sin\theta$
$\boldsymbol{R}\cos(\theta - \alpha)$
$R = \sqrt{8^2 + 15^2} = 17$
$\alpha = \tan^{-1}\left(\frac{15}{8}\right) = 61.9°$
$17\cos(\theta - 61.9°)$
Let $y = 17\cos(\theta - 61.9°)$
For y_{max}, $\cos(\theta - 61.9°) = 1$
$\Rightarrow y_{max} = 17$
$\theta - 61.9° = \cos^{-1}(1)$
$\theta - 61.9° = 0, 360°$
$\theta = 61.9°$

For y_{min}, $\cos(\theta - 61.9) = -1$
$\Rightarrow y_{min} = -17$
$\theta - 61.9° = \cos^{-1}(-1)$
$\theta - 61.9° = 180°$
$\theta = 241.9°$

(b) $4\sin\theta - 3\cos\theta$
$R = \sqrt{4^2 + 3^2} = 5$
$R\sin(\theta - \alpha)$
$5\sin(\theta - \alpha)$
$\alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$
$5\sin(\theta - 36.9°)$
Let $y = 5\sin(\theta - 36.9°)$
$y_{min} = -5$
$y_{max} = 5$
For y_{min}, $\sin(\theta - 36.9°) = -1$
$\theta - 36.9° = 270°$
$\theta = 306.9°$
For y_{max}, $\sin(\theta - 36.9°) = 1$
$\theta - 36.9° = 90°$
$\theta = 126.9°$

(c) $\boldsymbol{\sin\theta - 6\cos\theta}$
$R = \sqrt{1^2 + (-6)^2} = \sqrt{37}$
$\sqrt{37}\sin(\theta - \alpha)$
$\alpha = \tan^{-1}\left(\frac{6}{1}\right) = 80.5°$
$y = \sqrt{37}\sin(\theta - 80.1)$
$y_{max} = \sqrt{37}$ and it occurs when $\sin(\theta - 80.1) = 1$
$\theta - 80.1° = 90°$

$\theta = 170.5°$

$y_{min} = -\sqrt{37}$ and it occurs when

$\sin(\theta - 80.1) = -1$

$\theta - 80.1° = 270°$

$\theta = 350.5°$

(d) cos(θ + 60) – cosθ

$= \cos\theta\cos 60 - \sin\theta\sin 60 - \cos\theta$

$= \frac{1}{2}\cos\theta - \sin\theta\frac{\sqrt{3}}{2} - \cos\theta$

$= \frac{-1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$

$y = -\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right]$

$y = -[R\cos(\theta - \alpha)]$

$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = 1$

$y = -[\cos(\theta - \alpha)]$

$\alpha = \tan^{-1}(\sqrt{3}) = 60°$

$y = -[\cos(\theta - 60)]$

y_{min} occurs when $\cos(\theta - 60) = 1$

$\theta - 60° = 0, 360$

$\theta = 60°$

$y_{max} = 1$ and occurs when $\cos(\theta - 60°) = -1$

$\theta - 60° = \cos^{-1}(-1)$

$\theta = 240°$

Example VIII (UNEB Question)

Solve $\cos\theta + \sqrt{3}\sin\theta = 2$ for $0 \le \theta \le \pi$

Solution

$\cos\theta + \sqrt{3}\sin\theta = 2$

$R\cos(\theta - \alpha) = 2$

$R = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$2\cos(\theta - \alpha) = 2$

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

$2\cos(\theta - 60°) = 2$

$\cos(\theta - 60°) = 1$

$\theta - 60° = \cos^{-1}(1)$

$\theta - 60° = 0$

$\theta = 60°$

$\theta = \frac{\pi}{3}$

Since 180 = π radians, $\Rightarrow \theta = \frac{60\pi}{180} = \frac{\pi}{3}$

Example IX (UNEB Question)

(a) Express $4\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \beta)$, where R is a constant and β an acute angle.

Determine the maximum value of the expression and the value of θ for which it occurs

(b) Solve the equation $4\cos\theta - 5\sin\theta = 2.2$, for $0^0 < \theta < 360^0$.

Solution

$4\cos\theta - 5\sin\theta$

$R\cos(\theta + \beta)$

$\beta = \tan^{-1}\left(\frac{5}{4}\right) = 51.3°$

$R = \sqrt{4^2 + 5^2} = \sqrt{41}$

$\sqrt{41}\cos(\theta + 51.3°)$

Let $y = \sqrt{41}\cos(\theta + 51.3°)$

$y_{max} = \sqrt{41}$ and it occurs when $\cos(\theta + 51.3°) = 1$

$\theta + 51.3° = 0$

$\theta = -51.3°$

$\Rightarrow \theta = 308.7°$ $(0^0 < \theta < 360^0)$

$4\cos\theta - 5\sin\theta = 2.2$

$\Rightarrow \sqrt{41}\cos(\theta + 51.3°) = 2.2$

$\cos(\theta + 51.3°) = \frac{2.2}{\sqrt{41}}$

$\theta + 51.3° = 69.9°, 290.1°$

$\theta = 18.6°, 238.8°$

Example XI (UNEB Question)

Express $y = 8\cos x + 6\sin x$ in the form $R\cos(x - \alpha)$ where R is positive and α is acute . Hence find the maximum and minimum values of $\frac{1}{8\cos x + 6\sin x + 15}$

Solution

$8\cos x + 6\sin x = R\cos(x - \alpha)$

$8\cos x + 6\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

By comparison

$R\cos\alpha = 8$ …………..……………………… (i)

$R\sin\alpha = 6$……………...........……….(ii)

Eqn $(i)^2$ + Eqn $(ii)^2$;

$R^2 = 8^2 + 6^2 = 100$

$R = 10$

Eqn (ii) ÷ Eqn (i)

$\tan\alpha = \frac{6}{8}$

$\alpha = 36.87^0$

Hence $8\cos x + 6\sin x = 10\cos(x - 36.87^0)$

$Now \frac{1}{8\cos x + 6\sin x + 5} = \frac{1}{10\cos(x - 36.87) + 15}$

Note: For y to be maximum, the denominator must be minimum and for y to be minimum, the denominator must be maximum.

Let $m = \frac{1}{10\cos(x-36.87)+15}$

$$M_{\max} = \frac{1}{10\times(-1)+15}$$

$$= \frac{1}{-10+15} = \frac{1}{5} = 0.2$$

$$M_{\min} = \frac{1}{10\times 1+15}$$

$$= \frac{1}{25} = 0.04$$

The maximum and minimum values of $\frac{1}{8\cos x+(\sin x+15)}$ are 0.2 and 0.04 respectively.

Factor Formula

1. **sin P + sin Q =** $2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
2. **sin P − sin Q =** $2\cos(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$
3. **cos *P* + cos Q =** $2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$
4. **cos *P* – cos Q = -** $2\sin(\frac{P+Q}{2})\sin(\frac{P-Q}{2})$

Application of the factor formula

Example 1

Express the following in factors:

(a) $\sin 7\theta + \sin 5\theta$
(b) $\sin 4x - \sin 2x$
(c) $\cos 7x + \cos 5x$
(d) $\cos 3A - \cos 5A$
(e) $\sin(x + 30) + \sin(x - 30)$
(f) $\cos(x + 30) - \cos(x - 30)$
(g) $\cos \frac{3}{2}x - \cos \frac{x}{2}$
(h) $\frac{1}{2} + \cos 2\theta$
(i) $1 + \sin 2x$
(j) Sin $2(x + 40) + \sin 2(x - 40)$

Solution

(a) $\sin 7\theta + \sin 5\theta$

From $\sin P + \sin Q = 2\sin(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\sin 7\theta + \sin 5\theta = 2\sin(\frac{7\theta+5\theta}{2})\cos\frac{7\theta-5\theta}{2}$

$= 2\sin 6\theta\cos\theta$

(b) $\sin 4x - \sin 2x$

From $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\sin 4x - \sin 2x = 2\cos\frac{4x+2x}{2}\sin\frac{4x-2x}{2}$

$\sin 4x - \sin 2x = 2\cos 3x\sin x$

(c) $\cos 7x + \cos 5x$

From $\cos P + \cos Q = 2\cos(\frac{P+Q}{2})\cos(\frac{P-Q}{2})$

$\Rightarrow \cos 7x + \cos 5x = 2\cos(\frac{7x+5x}{2})\cos(\frac{7x-5x}{2})$

$= 2\cos 6x\cos x$

(d) $\cos 3A - \cos 5A$

From $\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\cos 3A - \cos 5A = -2\sin(\frac{3A+5A}{2})\sin(\frac{3A-5A}{2})$

$= -2\sin 4A\sin(-A)$

$= 2\sin 4A\sin A$

(e) $\sin(x + 30) + \sin(x - 30)$

$= -2\sin(\frac{(x+30)+(x-30)}{2})\cos(\frac{(x+30)-(x-30)}{2})$

$= -2\sin x\cos 30$

(f) $\cos(x + 30) - \cos(x - 30)$

$= 2\sin(\frac{(x+30)+(x-30)}{2})\sin(\frac{(x+30)-(x-30)}{2})$

$= 2\sin x\sin 30$

(g) $\cos(\frac{3x}{2}) - \cos\frac{x}{2} = -2\sin\left(\frac{\frac{3x}{2}+\frac{x}{2}}{2}\right)\sin\left(\frac{\frac{3x}{2}-\frac{x}{2}}{2}\right)$

$= 2\sin x\sin\frac{x}{2}$

(h) $\frac{1}{2} + \cos 2\theta$

$\cos 60 + \cos 2\theta$

$= 2\cos\frac{60+2\theta}{2}\cos\frac{60-2\theta}{2}$

$= 2\cos(30 + \theta)\cos(30 - \theta)$

(i) $1 + \sin 2x$

$\sin 90 + \sin 2x$

$2\sin\frac{90+2x}{2}\cos\frac{90-2x}{2}$

$= 2\sin(45 + x)\cos(45 - x)$

(j) $\sin 2(x + 40) + \sin 2(x - 40)$

$= 2\sin\frac{2(x+40)+2(x-40)}{2}\cos\frac{2(x+40)-2(x-40)}{2}$

$= 2\sin 2x\cos 80$

Example II

Solve the following equations from $x = 0°$ to $360°$ inclusive.

(a) $\cos x + \cos 5x = 0$
(b) $\sin 3x - \sin x = 0$
(c) $\sin(x + 10) + \sin x = 0$
(d) $\cos(2x + 10) + \cos(2x - 10) = 0$
(e) $\cos(x + 20) - \cos(x - 70) = 0$

Solution

(a) $\cos x + \cos 5x = 0$

$2\cos\frac{x+5x}{2}\cos\frac{x-5x}{2} = 0$

$2\cos 3x\cos{-2x} = 0$

$2\cos 3x\cos 2x = 0$

$\cos 3x \cos 2x = 0$

$\Rightarrow$ $\cos 2x = 0$ **OR**

$\cos 3x = 0$

For $\cos 2x = 0$;

$2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°, 810°$

$\Rightarrow$ $x = 45°, 135°, 225°, 315°$.

For $\cos 3x = 0$;

$3x = \cos^{-1}(0)$

$3x = 90°, 270°, 450°, 630°, 810°, 990°, 1170°$

$x = 30°, 90°, 150°, 210°, 270°, 330°$.

$\therefore$ The solutions to the equation $\cos x + \cos 5x = 0$ are 30°, 45°, 90°, 135°, 150°, 210°, 225°, 270°, 315°, 330°.

(b) $\sin 3x - \sin x = 0$

$2\cos \frac{3x+x}{2} \sin \frac{3x-x}{2} = 0$

$2\cos 2x \sin x = 0$

$\cos 2x \sin x = 0$

$\Rightarrow 2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°, 810°, 990°$

$\Rightarrow x = 45°, 135°, 225°, 315°$

And for $\sin x = 0$;

$x = \sin^{-1}(0)$

$x = 0, 180°, 360°$

$\Rightarrow$ The solutions to the equation sin 3x – sin x = 0 are 0, 45, 135, 180, 225, 315, 360.

(c) $\sin(x + 10) + \sin x = 0$

$2\sin\left(\frac{x+10+x}{2}\right)\cos\left(\frac{x+10-x}{2}\right) = 0$

$2\sin(x + 5) (\cos 5) = 0$

$\sin (x + 5) = 0$

$x + 5 = \sin^{-1}0)$

$x + 5 = 0, 180°, 360°$

$x = 355°, 175°$.

$\Rightarrow x = 175°, 335°$ are solutions to the equation $\sin(x + 10) + \sin x = 0$

(d) $\cos(2x + 10) + \cos(2x – 10) = 0$

$2\cos\left(\frac{(2x+10)+(2x-10)}{2}\right)\cos\left(\frac{(2x+10)-(2x-10)}{2}\right)$

$2\cos 2x \cos 10 = 0$

$\cos 2x = 0$

$2x = \cos^{-1}(0)$

$2x = 90°, 270°, 450°, 630°$.

$x = 45°, 135°, 225°, 315°$

$\Rightarrow$ The solutions to the equation $\cos(2x + 20) + \cos(2x – 10) = 0$ are $x = 45°, 135°, 225°$ and $315°$

(f) $\cos(x + 20) – \cos(x – 70) = 0$

$-2\sin \frac{(x+20)+(x-70)}{2} \sin \frac{(x+20)-(x-70)}{2} = 0$

$-2\sin(x – 25)\sin 45 = 0$

$\sin(x – 25) = 0$

$x – 25 = \sin^{-1}(0)$

$x – 25 = 0, 180°, 360°$

$x = 25, 205°, 385°$

Example II

Prove the following identities:

(a) $\dfrac{\cos B + \cos C}{\sin B - \sin C} = \cot \frac{B-C}{2}$

(b) $\dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan \frac{B-C}{2}$

(c) $\dfrac{\sin B - \sin C}{\sin B + \sin C} = \cot \frac{B+C}{2} \tan \frac{B-C}{2}$

(d) $\dfrac{\sin B + \sin C}{\cos B + \cos C} = \tan \frac{B+C}{2}$

Solution

(a) $\dfrac{\cos B + \cos C}{\sin B - \sin C}$

$= \dfrac{2\cos(\frac{B+C}{2})\cos(\frac{B-C)}{2})}{2\cos(\frac{B+C}{2})\sin(\frac{B-C)}{2})}$

$= \dfrac{\cos \frac{B-C}{2}}{\sin \frac{B-C}{2}}$

$= \cot(\frac{B-C}{2})$

(b) $\dfrac{\cos B - \cos C}{\sin B + \sin C}$

$= \dfrac{-2\sin \frac{B+C}{2} \sin \frac{B-C}{2}}{2\sin \frac{B+C}{2} \cos \frac{B-C}{2}}$

$= \dfrac{-\sin \frac{B-C}{2}}{\cos \frac{B-C}{2}}$

$= -\tan(\frac{B-C}{2})$

$\Rightarrow \dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan(\frac{B-C}{2})$

(c) $\dfrac{\sin B - \sin C}{\sin B + \sin C}$

$\dfrac{\sin B - \sin C}{\sin B + \sin C} = \dfrac{2\cos(\frac{B+C}{2})\sin(\frac{B-C}{2})}{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}$

$= \dfrac{\cos(\frac{B+C}{2})}{\sin(\frac{B+C}{2})} \times \dfrac{\sin(\frac{B-C}{2})}{\cos(\frac{B-C}{2})}$

$= \cot(\frac{B+C}{2})\tan(\frac{B-C}{2})$

(d) $\dfrac{\sin B + \sin C}{\cos B + \cos C}$

$= \dfrac{2\sin \frac{B+C}{2} \cos \frac{B-C}{2}}{2\cos \frac{B+C}{2} \cos \frac{B-C}{2}}$

$= \tan \frac{B-C}{2}$

$\Rightarrow \dfrac{\sin B + \sin C}{\cos B + \cos C} == \tan \frac{B-C}{2}$

Example IV

Prove the following

(a) $\sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x = \sin 2x(2\sin x + 1)$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin\theta(\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2\theta \cos 2\theta$

Solutions

(a) $\sin x + \sin 2x + \sin 3x$

$= \sin x + \sin 3x + \sin 2x$

$= 2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2} + \sin 2x$

$= 2\sin 2x\cos(-x) + \sin 2x$

$= 2\sin 2x\cos x + \sin 2x$

$= \sin 2x(2\cos x + 1)$

$\Rightarrow \sin x + \sin 2x + \sin 3x = \sin 2x(2\cos x + 1)$

(b) $\cos x + \sin 2x - \cos 3x$

$= \cos x - \cos 3x + \sin 2x$

$= -2\sin\frac{x+3x}{2}\sin\frac{x-3x}{2} + \sin 2x$

$= -2\sin 2x\sin(-x) + \sin 2x$

$= 2\sin 2x\sin x + \sin 2x$

$= \sin 2x[2\sin x + 1]$

$\Rightarrow \cos x + \sin 2x - \cos 3x = \sin 2x[2\sin x + 1]$

(c) $\cos\theta - 2\cos 3\theta + \cos 5\theta$

$= \cos\theta - \cos 3\theta + \cos 5\theta - \cos 3\theta$

$= -2\sin 2\theta\sin(-\theta) + {}^{-}2\sin 4\theta\sin\theta$

$= 2\sin 2\theta\sin\theta - 2\sin 4\theta\sin\theta$

$= 2\sin\theta(\sin 2\theta - \sin 4\theta)$

$\Rightarrow \cos\theta - 2\cos 3\theta + \cos 5\theta = 2\sin\theta(\sin 2\theta - \sin 4\theta)$

(d) $\sin x - \sin(x + 60) + \sin(x + 120)$

$= \sin x + \sin(x + 120) - \sin(x + 60)$

$= 2\sin(x + 60)\cos -60 - \sin(x + 60)$

$= \sin(x + 60) - \sin(x + 60)$

$= 0$

$\Rightarrow \sin x - \sin(x + 60) + \sin(x + 120) = 0$

(e) $1 + 2\cos 2\theta + \cos 4\theta$

Since $\cos 4\theta = \cos^2 2\theta - 1$,

$\Rightarrow 1 + 2\cos 2\theta + 2\cos^2 2\theta - 1$

$= 2\cos 2\theta + 2\cos^2 2\theta$

$= 2\cos 2\theta[1 + \cos 2\theta]$

$= 2\cos 2\theta[1 + 2\cos^2\theta - 1]$

$= 4\cos^2\theta\cos 2\theta$

$\Rightarrow 1 + 2\cos 2\theta + \cos 4\theta = 4\cos^2\theta\cos 2\theta$

Example V

Solve the following equations for values of θ from 0° to 180° inclusive

(a) $\cos\theta + \cos 3\theta + \cos 5\theta = 0$

(b) $\sin\theta - 2\sin 2\theta + \sin 3\theta = 0$

(c) $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

(e) $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Solution

(a) $\cos\theta + \cos 3\theta + \cos 5\theta = 0$

$\cos\theta + \cos 5\theta + \cos 3\theta = 0$

$2\cos 3\theta\cos -2\theta + \cos 3\theta = 0$

$\cos 3\theta(2\cos 2\theta + 1] = 0$

Either $\cos 3\theta = 0$ **OR**

$\cos 2\theta = -\frac{1}{2}$

For $\cos 3\theta = 0$;

$3\theta = \cos^{-1}(0)$

$3\theta = 90°, 270°, 450°, 630°, 810°, 990°$

$\theta = 30°, 90°, 150°, 210°, 270, 330°$

$\Rightarrow \theta = 30°, 90°, 150°$ (for $0° \le \theta \le 180°$)

For $\cos 2\theta = -\frac{1}{2}$;

$2\theta = \cos^{-1}(\frac{-1}{2})$

$2\theta = 120°, 240°$.

$\theta = 60°, 120°$

$\Rightarrow$ 30°, 60°, 90°, 120°, 150° are the solutions to the equation $\cos 3\theta + \cos 3\theta + \cos 5\theta = 0$

(b) $\sin\theta - 2\sin 2\theta + \sin 3\theta = 0$

$\sin\theta + \sin 3\theta - 2\sin 2\theta = 0$

$2\sin 2\theta\cos(-\theta) - 2\sin 2\theta = 0$

$2\sin 2\theta\cos\theta - 2\sin 2\theta = 0$

$2\sin 2\theta(\cos\theta - 1) = 0$

Either $\sin 2\theta = 0$ **OR** $\cos\theta = 1$

For $\sin 2\theta = 0$;

$2\theta = \sin^{-1}0$

$2\theta = 0°, 180°, 360°$

$\Rightarrow \theta = 0°, 90°, 180°$

(c) $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

$\sin\theta - \sin 3\theta + \cos 2\theta = 0$

$2\cos 2\theta\sin -\theta + \cos 2\theta = 0$

$\cos 2\theta(-2\sin\theta + 1) = 0$

$\cos 2\theta = 0$ **OR** $\sin\theta = \frac{1}{2}$

For $\cos 2\theta = 0$

$2\theta = \cos^{-1}0$

$2\theta = 90°, 270°, 450°$

$= 45°, 135°$

For $\sin\theta = \frac{1}{2}$;

$\theta = \sin^{-1}(\frac{1}{2})$

$\theta = 30°, 150°$

$\Rightarrow$ 30°, 45°, 135°, 150° are the solutions to the equation $\sin\theta + \cos 2\theta - \sin 3\theta = 0$

(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

$(\sin 2\theta + \sin 6\theta) + \sin 4\theta = 0$

$2\sin 4\theta \cos -2\theta + \sin 4\theta = 0$

$2\sin 4\theta \cos 2\theta + \sin 4\theta = 0$

$\sin 4\theta (2\cos 2\theta + 1) = 0$

For $\sin 4\theta = 0$;

$4\theta = \sin^{-1}0$

$4\theta = 0, 180, 360, 540, 720$

$= 0, 45, 90, 135, 180$

For $2\cos 2\theta + 1 = 0$

$\cos 2\theta = -\frac{1}{2}$

$2\theta = 120°, 240°$

$\theta = 60°, 120°$

$\Rightarrow$ 0°, 45°, 60°, 90°, 120°, 135°, 180° are the solutions to the equation $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$

(e) $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

$\cos\frac{1}{2}\theta + \cos\frac{5}{2}\theta + 2\cos\frac{3}{2}\theta = 0$

$2\cos\frac{6\theta}{2}\cos(-\theta) + 2\cos\frac{3\theta}{2} = 0$

$2\cos\frac{3\theta}{2}(\cos\theta + 1) = 0$

$\cos\frac{3\theta}{2} = 0$

$\frac{3\theta}{2} = \cos^{-1}(0)$

$\frac{3\theta}{2} = 90, 270, 450$

$\theta = 60, 180$

For $(\cos\theta + 1) = 0$;

$\cos\theta = {}^{-}1$

$\theta = \cos^{-1}({}^{-}1)$

$\theta = 180$

$\Rightarrow$ 60, 180 are the solutions to the equation $\cos\frac{1}{2}\theta + 2\cos\frac{3}{2}\theta + \cos\frac{5}{2}\theta = 0$

Example V

Prove the following identities if A, B and C are taken to be angles of a triangle.

(a) $\sin A + \sin(B - C) = 2\sin B \cos C$

(b) $\cos A - \cos(B - C) = -2\cos B \cos C$

(c) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(d) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(e) $\cos A + \cos B + \cos C - 1$

$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solutions

$\sin A + \sin(B - C)$

$= 2\sin\frac{A+B-C}{2}\cos\frac{A-(B-C)}{2}$

$= 2\sin\frac{A+B-C}{2}\cos\frac{A+C-B}{2}$

But $A + B + C = 180$

$A + B + C - 2C = 180 - 2C$

$A + B - C = 180 - 2C$

$\frac{A+B-C}{2} = 90 - C$

$\Rightarrow \sin\frac{A+B-C}{2} = \sin(90 - C)$

$= \sin 90 \cos C - \cos 90 \sin C$

$= \cos C$

$A + B + C = 180$

$A + C + B - 2B = 180 - 2B$

$A + C - B = 180 - 2B$

$\cos\frac{A+C-B}{2} = \cos\frac{180-2B}{2}$

$\cos\frac{A+C-B}{2} = \cos(90 - B)$

$= \cos 90 \cos B + \sin 90 \sin B$

$= \sin B$

$\Rightarrow \sin A + \sin(B - C) = 2\sin B \cos C$

(c) $\sin A + \sin B + \sin C$

$= \sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

But $A + B + C = 180$

$C = 180 - (A + B)$

$\frac{C}{2} = \sin(90 - \frac{A+B}{2})$

$\sin\frac{C}{2} = \sin 90 \cos\frac{A+B}{2} - \cos 90 \sin\frac{A+B}{2}$

$= \cos\frac{A+B}{2}$

$\cos\frac{C}{2} = \cos(90 - \frac{A+B}{2})$

$\cos\frac{C}{2} = \cos 90 \cos\frac{A+B}{2} + \sin 90 \sin\frac{A+B}{2}$

$= \sin\frac{A+B}{2}$

$\Rightarrow 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{A+B}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$

$= 2\cos\frac{C}{2}[2\cos\frac{A}{2}\cos\frac{B}{2}]$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(d) $\sin 2A + \sin 2B + \sin 2C$.

$= 2\sin(A + B)\cos(A - B) + 2\sin C \cos C$

But $A + B + C = 180$

$C = 180 - (A + B)$

$\Rightarrow \sin C = \sin[180 - (A + B)]$

$\sin C = \sin 180 \cos (A + B) - \cos 180 \sin(A + B)$

$\sin C = \sin(A + B)$

$\cos C = \cos(180 - (A+B))$

$\cos C = \cos 180 \cos(A + B) + \sin 180 \sin(A + B)$

$= -\cos(A + B)$

$\Rightarrow 2\sin(A + B)\cos(A - B) + 2\sin C \cos C$

$= 2\sin C \cos (A - B) + 2\sin C(-\cos(A+B))$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$= 2\sin C[-2\sin A \sin -B]$

$= 4\sin A \sin B \sin C$

$\Rightarrow \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(e) $\cos A + \cos B + \cos C - 1$

$\cos C = 2\cos^2 \frac{C}{2} - 1$

$\cos C = 1 - 2\sin^2 \frac{C}{2}$

$\Rightarrow 2\sin^2 \frac{C}{2} = 1 - \cos C$

$\cos A + \cos B + \cos C - 1 = \cos A + \cos B - 2\sin^2 \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin^2 \frac{C}{2}$

$A + B + C = 180$

$\frac{C}{2} = 90 - \frac{A+B}{2}$

$\sin \frac{C}{2} = \sin (90 - \frac{A+B}{2})$

$\sin \frac{C}{2} = \sin 90 \cos \frac{A+B}{2} - \cos 90 \sin \frac{A+B}{2}$

$\Rightarrow \sin \frac{C}{2} = \cos \frac{A+B}{2}$

$\Rightarrow 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \sin \frac{C}{2}$

$= 2\cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2\sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{A+B}{2}$

$= 2\sin \frac{C}{2} [\cos \frac{A-B}{2} - \cos \frac{A+B}{2}]$

$= 2\sin \frac{C}{2} [-2\sin \frac{A}{2} \sin \frac{-B}{2}]$

$= 2\sin \frac{C}{2} [2\sin \frac{A}{2} \sin \frac{B}{2}]$

$= 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$\Rightarrow \cos A + \cos B + \cos C - 1 = 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Example VI (UNEB 2007)

Show that $\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta} = -\tan^2 \frac{\theta}{2}$

Solution

$$\frac{\sin\theta - 2\sin 2\theta + \sin 3\theta}{\sin\theta + 2\sin 2\theta + \sin 3\theta}$$

$$= \frac{Sin3\theta + \sin\theta - 2\sin 2\theta}{\sin 3\theta + \sin\theta + 2\sin 2\theta}$$

$$= \frac{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) - 2\sin 2\theta}{2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + 2\sin 2\theta}$$

$$= \frac{2\sin 2\theta\cos\theta - 2\sin 2\theta}{2\sin 2\theta\cos\theta + 2\sin 2\theta}$$

$$= \frac{2\sin 2\theta(\cos\theta - 1)}{2\sin 2\theta(\cos\theta + 1)}$$

$$= \frac{\cos\theta - 1}{\cos + 1} = -\frac{-(1-\cos\theta)}{1+\cos\theta}$$

But $\cos\theta = 1 - 2\sin^2 \frac{\theta}{2}$

$\cos\theta = 2\sin^2 \frac{\theta}{2} - 1$

$$-\frac{-(1-\cos\theta)}{1+\cos\theta} = \frac{-(1-(1-2\sin^2 \frac{\theta}{2}))}{1-2\sin^2 \frac{\theta}{2}}$$

$$= \frac{-2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}$$

$$= -\tan^2 \frac{\theta}{2}$$

Example VII (UNEB Question)

Show that $\frac{\sin 3\theta\sin 6\theta + \sin\theta\sin 2\theta}{\sin 3\theta\cos 6\theta + \sin\theta\cos 2\theta} = \tan 5\theta$.

Solution

$$= \frac{\sin 6\theta\sin 3\theta + \sin 2\theta\sin\theta}{\cos 6\theta\sin 3\theta + \cos 2\theta\sin\theta}$$

$\cos A - \cos B = -\sin \frac{A+B}{2} \sin \frac{A-B}{2}$

$\sin \frac{A+B}{2} \sin \frac{A-B}{2} = \frac{-1}{2}(\cos A - \cos B)$

$\sin 6\theta\sin 3\theta = \frac{-1}{2}(\cos A - \cos B)$

$\frac{A+B}{2} = 6\theta$

$A + B = 6\theta$ …………………….. (i)

$\frac{A-B}{2} = 3\theta$

$A - B = 6\theta$ ……………………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously;

$A = 9\theta$, $B = 3\theta$

$$\sin 6\theta\sin 3\theta = \frac{-1}{2}(\cos 9\theta - \cos 3\theta)$$

$$\Rightarrow \frac{\sin 3\theta\sin 6\theta + \sin\theta\sin 2\theta}{\sin 3\theta\cos 6\theta + \sin\theta\cos 2\theta}$$

$$= \frac{\frac{-1}{2}(\cos 9\theta - \cos 3\theta) + \frac{-1}{2}(\cos 3\theta - \cos\theta)}{\frac{1}{2}(\sin 9\theta - \sin 3\theta) + \frac{1}{2}(\sin 3\theta - \sin\theta)}$$

$$= \frac{\frac{1}{2}(\cos\theta - \cos 9\theta)}{\frac{1}{2}(\sin 9\theta - \sin\theta)}$$

$$\frac{-2\sin 5\theta \sin(-4\theta)}{2\cos 5\theta \sin 4\theta}$$

$$\Rightarrow \frac{2\sin 5\theta \sin(4\theta)}{2\cos 5\theta \sin 4\theta} = \tan 5\theta$$

Example VIII (UNEB Question)

If A, B, C are angles of the triangle, show that
$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$.

Solution

cos 2A + cos 2B + cos 2C

$2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1$

$= -1 + 2\cos(A + B)\cos(A - B) + 2\cos^2 C$

A + B + C = 180

A + B = (180 – C)

cos(A + B) = cos(180 – C)

cos (A + B) = cos 180 cos C + sin 180 sin C

= -cos C

$\Rightarrow$ $-1 + 2\cos(A + B)\cos(A - B) + 2\cos^2 A$

$= -1 - 2\cos C\cos(A - B) + 2\cos^2 C$

= -1 – 2cos C[cos(A – B) − cos C]

= -1 – 2cos C[cos(A – B) – cos C]

$\cos C = -\cos(A + B)$

= -1 – 2cos C[cos(A – B) + cos (A + B)]

= -1 – 4cos A cos B cos C.

$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$.

Example IX (UNEB Question)

Use the factor formula to show that

$$\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$$

Solution

$$\frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A}$$

$$= \frac{2\sin(A+B)\cos B}{2\cos(A+B)\cos B}$$

$$= \frac{\sin(A+B)}{\cos(A+B)}$$

= tan(A + B)

$$\Rightarrow \frac{\sin(A+2B)+\sin A}{\cos(A+2B)+\cos A} = \tan(A+B)$$

UNEB 2008

(i) Prove that $\frac{\cos A + \cos B}{\sin A + \sin B} = \cot\frac{A+B}{2}$

(ii) Deduce that $\frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$ where A, B and C are

solution

(i) $$\frac{\cos A + \cos B}{\sin A + \sin B} = \frac{2\cos\frac{A+B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$= \frac{2\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}}$$

$$= \cot\frac{A+B}{2}$$

(ii) $A + B + C = 180^0$

A + B =180 – C

$$\frac{A+B}{2} = 90 - \frac{C}{2}$$

$$\cot\frac{A+B}{2} = \frac{\cos\left(90-\frac{c}{2}\right)}{\sin\left(90-\frac{c}{2}\right)}$$

$$= \frac{\cos 90\cos\frac{c}{2} + \sin 90\sin\frac{c}{2}}{\sin 90\cos\frac{c}{2} - \cos 90\sin\frac{c}{2}}$$

$$= \frac{\sin\frac{c}{2}}{\cos\frac{c}{2}}$$

$$= \tan\frac{c}{2}$$

$$\Rightarrow \frac{\cos A + \cos B}{\sin A + \sin B} = \tan\frac{C}{2}$$

Example X (UNEB Question)

Solve $\sin x - \sin 4x = \sin 2x - \sin 3x$ for $-\pi \le x \le \pi$

Solution

$\sin x - \sin 4x = \sin 2x - \sin 3x$

$\sin 3x + \sin x = \sin 4x + \sin 2x$

$$2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 2\sin\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)$$

$$2\sin(2x)\cos x = 2\sin 3x\cos x$$

$$\sin 2x\cos x - \sin 3x\cos x = 0$$

$$\cos x(\sin 2x - \sin 3x) = 0$$

Taking cos = 0

$$x = \cos^{-1}(0)$$

$$x = \frac{-\pi}{2}, \frac{\pi}{2}$$

Taking $\sin 2x - \sin 3x = 0$

$\sin 3x - \sin 2x = 0$

$$2\cos\left(\frac{3x+2x}{2}\right)\sin\left(\frac{3x-2x}{2}\right) = 0$$

$\cos\left(\frac{5}{2}x\right)\sin\left(\frac{1}{2}x\right) = 0$

Either $\cos\left(\frac{5}{2}x\right) = 0$

$\frac{5}{2}x = \cos^{-1}(0)$

$\frac{5}{2}x = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi$

$x = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$

Or $\quad \text{Sin}\left(\frac{1}{2}x\right) = 0$

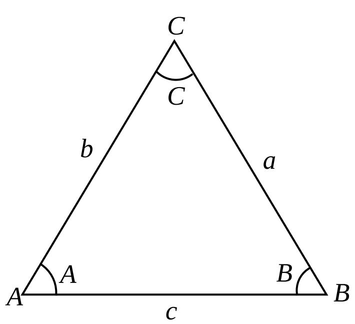
$\frac{1}{2}x = \sin^{-1}(0) = 0, \pm\pi$

$x = 0^0$

$\Rightarrow x = 0, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{-3\pi}{5}$ are the solutions to the equation

Relationship between sides of a triangle

In a triangle ABC with angles A, B and C, we denote the side opposite these angles by their corresponding small letters a, b, and c respectively as shown in the figure below.



The sine rule

Let O be the centre of the circle circumscribing the triangle ABC with radius, *R*.

Figure I

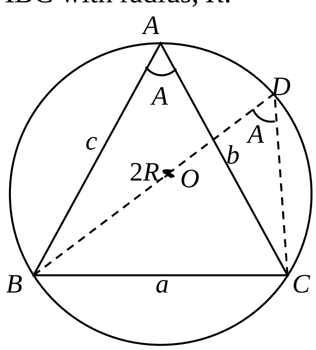


Figure II

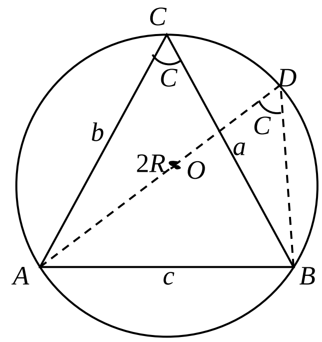
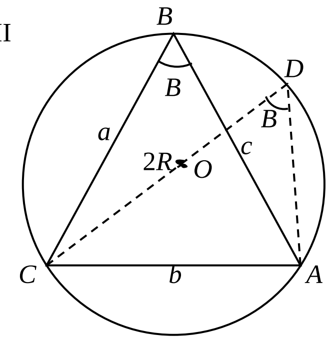


Figure III



From figure I, $\angle$BCD = 90°

Since this angle is subtended by the diameter,

$\Rightarrow \sin A = \frac{a}{2R}$ from figure I.

$\Rightarrow 2R = \frac{a}{\sin A}$ ………………… (i)

From figure II;

$\sin C = \frac{c}{2R}$

$2R = \frac{c}{\sin C}$ …………………… (ii)

From figure III;

$\sin B = \frac{b}{2R}$

$2R = \frac{b}{\sin B}$ …………………….. (iii)

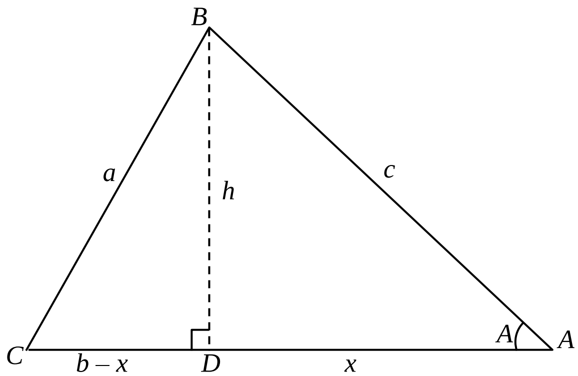
Equating equations (i), (ii), and (iii)

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

This is the sine rule

The Cosine rule

Consider a triangle ABC. Assume angle A is acute.



Considering the right-angled triangle BDA,

$x^2 + h^2 = c^2$ ……………………..…. (i)

from the right-angled triangle BCD,

$a^2 = (b - x)^2 + h^2$

$a^2 = b^2 - 2bx + x^2 + h^2$ ………….. (ii)

From Eqn (i);

$h^2 = c^2 - x^2$ ……………………..…. (iii)

Substituting Eqn (iii) in Eqn (ii)

$a^2 = b^2 - 2bx + x^2 + c^2 - x^2$

$a^2 = b^2 - 2bx + c^2$ ……………….. (iv)

From triangle ABD;

$$\cos A = \frac{x}{c}$$

$x = c \cos A$ …………..………….. (v)

Substituting Eqn (v) into (iv)

$\Rightarrow a^2 = b^2 - 2bc \cos A + c^2$

$a^2 = b^2 + c^2 - 2bc \cos A$

Application of cosine and sine rules

Example I

Prove that in a triangle ABC, $\dfrac{a^2 - b^2}{c^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$

Solution

From the sine rule; $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$

$$\frac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 C}$$

$$\frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\frac{(\sin A + \sin B)(\sin(A - \sin B)}{\sin C \sin C}$$

$A + B + C = 180$

$C = 180 - (A + B)$

$\sin C = \sin(180 - (A + B))$

$\sin C = \sin 180 \cos (A+B) - \cos 180 \sin (A+B)$

$= \sin (A + B)$

$$\frac{(\sin A + \sin B)(\sin A - \sin B)}{[\sin(A+B)]^2}$$

$$= \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cdot 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\left[2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\right]\left[\sin(A+B)\right]}$$

$$= \frac{2\sin\frac{A-B}{2}\cos\frac{A-B}{2}}{\sin(A+B)}$$

$$= \frac{\sin(A-B)}{\sin(A+B)}$$

Example II

Prove that in any triangle ABC, $\dfrac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \tan B \cot C$

Solution

From the cosine rule;

$a^2 = b^2 + c^2 - 2bc \cos A$ ……………… (i)

$b^2 = a^2 + c^2 - 2ac \cos B$ ……………… (ii)

$c^2 = a^2 + b^2 - 2ab \cos C$ ……………… (iii)

From Eqn (i);

$2ac \cos B = a^2 + c^2 - b^2$

From Eqn (iii);

$2ab \cos C = a^2 + b^2 - c^2$

$$\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \frac{2ab\cos C}{2ac\cos B}$$

$$= \frac{b\cos C}{c\cos B}$$

But from the sine rule;

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$

$$\Rightarrow \frac{b\cos C}{c\cos B} = \frac{2R\sin B\cos C}{2R\sin C\cos B}$$

$$= \frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C}$$

$= \tan B \times \cot C$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{a^2 - b^2 + c^2} = \tan B \cot C$$

Example III

Prove that in any triangle ABC, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

Solution

$\frac{b-c}{b+c}\cot\frac{A}{2}$

From the sine rule; $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$

$$\Rightarrow \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}\cot\frac{A}{2} = \frac{\sin B - \sin C}{\sin B + \sin C}\cot\frac{A}{2}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\sin\frac{A}{2}}$$

But $A + B + C = 180$

$A = 180 - (B + C)$

$$\frac{A}{2} = 90 - \frac{B+C}{2}$$

$\cos\frac{A}{2} = \cos(90 - \frac{B+C}{2})$

$= \cos 90 \cos\frac{B+C}{2} + \sin 90 \sin\frac{B+C}{2}$

$= \sin\frac{B+C}{2}$

$\sin\frac{A}{2} = \sin(90 - \frac{B+C}{2})$

$= \sin 90 \cos\frac{B+C}{2} - \cos 90 \sin\frac{B+C}{2}$

$= \cos\frac{B+C}{2}$

$$\Rightarrow \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\sin\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cos\frac{B+C}{2}} = \tan\frac{B-C}{2}$$

$$\Rightarrow \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Example IV

Prove that in any triangle *ABC*,

$$\frac{bc}{ab+ac} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

Solution

From the sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$

$$\frac{bc}{ab+ac} = \frac{2R\sin B \cdot 2R\sin C}{(2R\sin A)(2R\sin B) + (2R\sin A)(2R\sin C)}$$

$$= \frac{4R^2\sin B\sin C}{4R^2\sin A\sin B + 4R^2\sin A\sin C}$$

$$= \frac{\sin B\sin C}{\sin A\sin B + \sin A\sin C}$$

$$= \frac{\frac{\sin B\sin C}{\sin B\sin C}}{\frac{\sin A\sin B}{\sin B\sin C} + \frac{\sin A\sin C}{\sin B\sin C}}$$

$$= \frac{1}{\frac{\sin A}{\sin C} + \frac{\sin A}{\sin B}}$$

$$= \frac{1}{\sin A(\frac{1}{\sin C} + \frac{1}{\sin B})}$$

$$= \frac{1}{\sin A(\text{cosec}B + \text{cosec }C)}$$

$$= \frac{1}{\sin A} \times \frac{1}{(\text{cosec }B + \text{cosec }C)}$$

From triangle ABC;

$A + B + C = 180$

$A = 180 - (B + C)$

$\sin A = \sin(180 - (B + C))$

$= \sin 180 \cos B + C - \cos 180 \sin(B + C)$

$= \sin (B + C)$

$$\Rightarrow \frac{1}{\sin(B+C)} \times \frac{1}{(\text{cosec }B + \text{cosec }C)} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

$$\Rightarrow \frac{bc}{ab+ac} = \frac{\text{cosec }(B+C)}{\text{cosec}B + \text{cosec }C}$$

Area of a triangle

Let D denote the area of a triangle ABC, then

$$D = \frac{1}{2}bc\sin A$$

$$\Rightarrow D = \frac{1}{2}bc \cdot 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$D = bc \sin\frac{A}{2} \cos\frac{A}{2}$

$$S = \frac{a+b+c}{2}$$

Where S is the semi perimeter.

From the cosine rule, $a^2 = b^2 + c^2 - 2bc \cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$\cos A = 1 - 2\sin^2\frac{A}{2}$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\sin^2\frac{A}{2} = \frac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$$

$$\sin^2\frac{A}{2} = \frac{1}{2}\left(\frac{a^2-b^2-c^2+2bc}{2bc}\right)$$

$$\sin^2\frac{A}{2} = = \frac{1}{2}\left(\frac{a^2-(b-c)^2}{2bc}\right)$$

$$\sin\frac{A}{2} = \sqrt{\frac{(a+b-c)(a+c-b)}{4bc}}$$

$a + b + c = 2s$

$a + b - c = a + b + c - 2c$

$= 2s - 2c$

$= 2(s - c)$

$a + c - b = a + b + c - 2b$

$= 2s - 2b$

$= 2(s - b)$

$$\sin\frac{A}{2} = \sqrt{\frac{2(s-c)2(s-b)}{4bc}}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

From the cosine rule, $a^2 = b^2 + c^2 - 2bc \cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$\cos A = 2\cos^2 \frac{A}{2} - 1$

$\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A)$

$$\cos^2 \frac{A}{2} = \frac{1}{2}\left(\frac{2bc + b^2 + c^2 - a^2}{2bc}\right)$$

$$\cos^2 \frac{A}{2} = \frac{1}{2}\left(\frac{(b+c)^2 - a^2}{2bc}\right)$$

$$\cos^2 \frac{A}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\cos \frac{A}{2} = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}}$$

$a + b + c = 2s$

$b + c - a = a + b + c - 2a$

$= 2s - 2a$

$= 2(s - a)$

$$\cos \frac{A}{2} = \sqrt{\frac{2s \cdot 2(s-a)}{4bc}} = \sqrt{\frac{s(s-a)}{bc}}$$

From the area of a triangle D;

$D = bc \sin \frac{A}{2} \cos \frac{A}{2}$

$$\cos \frac{A}{2} = bc\sqrt{\frac{(S-b)(S-c}{bc}} \cdot \sqrt{\frac{S(S-a)}{bc}}$$

$$= bc\frac{\sqrt{S(S-a)(s-b)(S-c)}}{bc}$$

$$= \sqrt{S(S-a)(S-b)(S-c)}$$

The area of a triangle is $\sqrt{S(S-a)(S-b)(S-c)}$

This is called the Heron formula named after the Greek Mathematician Heron

Differentiation and integration of trigonometric functions

Function	Differentiate	Integrate
Sin x	Cos x	$-\cos x$
cos x	-sinx	sin x
Sin ax	a cos ax	$\frac{-1}{a}\cos ax$
cos ax	$-a \sin ax$	$\frac{1}{a}\sin ax$
sin $3x$	$3\cos 3x$	$\frac{-1}{3}\cos 3x$
cos $3x$	$-3\sin 3x$	$\frac{1}{3}\sin 3x$

Differentiation of trigonometric functions

Example I

Differentiate the following

(a) $\sin 6x$

(b) $-3\cos 5x$

(c) $-4\sin \frac{3}{2}x$

(d) $\sin x^2$

(e) $2\sin \frac{1}{2}(x + 1)$

Solutions

(a) $y = \sin 6x$

$$\frac{dy}{dx} = 6\cos 6x$$

(b) $-3\cos 5x$

$y = -3\cos 5x$

$$\frac{dy}{dx} = -3[5(-\sin 5x)]$$

$$\frac{dy}{dx} = 15\sin 5x$$

(c) $-4\sin \frac{3}{2}x$

$y = -4\sin \frac{3}{2}x$

$$\frac{dy}{dx} = -4 \times \frac{3}{2}\cos \frac{3x}{2}$$

$$= -6\cos \frac{3x}{2}$$

(d) $\sin x^2$

$y = \sin x^2$

$$\frac{dy}{dx} = 2x\cos x^2$$

(e) $2\sin \frac{1}{2}(x + 1)$

$y = 2\sin \frac{1}{2}(x + 1)$

$$\frac{dy}{dx} = 2 \times \frac{1}{2}\cos \frac{1}{2}(x+1)$$

$$\frac{dy}{dx} = \cos \frac{1}{2}(x+1)$$

Example II

Differentiate the following

(a) $\sin^2 x$
(b) $4\cos^2 x$
(c) $\cos^3 x$
(d) $2\sin^3 x$
(e) $3\sin^4 2x$
(f) $\sqrt{\sin 2x}$

Solutions

(a) $\sin^2 x$

$y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x(\cos x)$$

(b) $4\cos^2 x$

$y = 4\cos^2 x$

$$\frac{dy}{dx} = 8\cos x(-\sin x)$$

$$= -8\sin x\cos x$$

$$\frac{dy}{dx} = -8\sin x\cos x$$

(c) $\cos^3 x$

$y = \cos^3 x$

$$\frac{dy}{dx} = 3(\cos^2 x)(-\sin x)$$

$$\frac{dy}{dx} = -3\cos^2 x\sin x$$

(d) $2\sin^3 x$

$y = 2\sin^3 x$

$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$

$$\frac{dy}{dx} = 6\sin^2 x(\cos x)$$

(e) $3\sin^4 2x$

$y = 3\sin^4 2x$

$$\frac{dy}{dx} = 12\sin^3 2x(2\cos 2x)$$

$$\frac{dy}{dx} = 24\sin^3 2x\cos 2x$$

(f) $\sqrt{\sin 2x}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin 2x)^{\frac{-1}{2}} \cdot 2\cos 2x$$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Example II

Differentiate the following

(a) $x\cos x$
(b) $\boldsymbol{x}\sin 2x$
(c) $x^2\sin x$
(d) $\frac{x}{\sin x}$
(e) $\frac{x^2}{\cos x}$
(f) $\frac{\cos 2x}{x}$

Solutions

(a) $y = x\cos x$

From $y = uv$;

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = \boldsymbol{x}(-\sin x) + \cos x$$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

(b) $\boldsymbol{x}\sin 2x$

$y = x\sin 2x$

$$\frac{dy}{dx} = x\,.2\cos 2x + \sin 2x.1$$

$$\frac{dy}{dx} = 2x\cos 2x + \sin 2x$$

(c) $x^2\sin x$

$y = x^2\sin x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^2\cos x + (\sin x)\,2x$$

$$\frac{dy}{dx} = x^2\cos x + 2x\sin x$$

(d) $\frac{x}{\sin x}$

$$y = \frac{x}{\sin x}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sin x \cdot 1 - x\cos x}{(\sin x)^2}$$

$$\frac{dy}{dx} = \frac{\sin x - x\cos x}{(\sin x)^2}$$

(e) $\frac{x^2}{\cos x}$

$$y = \frac{x^2}{\cos x}$$

From $y = \frac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 2x - x^2(-\sin x)}{(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x^2}$$

(f) $\frac{\cos 2x}{x}$

$$y = \frac{\cos 2x}{x}$$

From $y = \frac{u}{v}$;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x \cdot {}^{-}2\sin 2x - \cos 2x}{x^2}$$

$$\frac{dy}{dx} = \frac{{}^{-}2x\sin 2x - \cos 2x}{x^2}$$

Derivatives of tan x, cot x, sec x, and cosec x

(i) $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x)$

$\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

Proofs

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$= -\text{cosec}^2 x$

$\frac{d}{dx} = -\text{cosec}^2 x$

(iii) $\frac{d}{dx}$ **(sec x)**

$$= \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$\frac{dy}{dx} = \frac{\cos x \cdot 0 - 1({}^{-}\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$= \sec x \tan x$

$$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\text{cosec } x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot 0 - 1 \cdot \cos x)}{(\sin x)^2}$$

$$\frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$= -\cot x \text{ cosec } x$

Example I

Differentiate the following

(a) $\tan 2x$

(b) $\cot 3x$

(c) $2\text{cosec}\frac{1}{2}x$

(d) $-\tan(2x+1)$

(e) $\frac{1}{3}\sec(3x-2)$

(f) $\tan\sqrt{x}$

Solution

(a) $\tan 2x$

$y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

(b) $\cot 3x$

$y = \cot 3x$

$\frac{dy}{dx} = 3(-\text{cosec}^2 3x)$

$= -3\text{cosec}^2 3x$

(c) $2\text{cosec}\,\tfrac{1}{2}x$

$y = 2\text{cosec}\,\tfrac{1}{2}x$

$\frac{dy}{dx} = 2\cdot\frac{1}{2}\left(-\text{cosec}\,\tfrac{1}{2}x\cot\tfrac{1}{2}x\right)$

$\frac{dy}{dx} = \left(-\text{cosec}\,\tfrac{1}{2}x\cot\tfrac{1}{2}x\right)$

(d) $-\tan(2x+1)$

$\boldsymbol{y} = -\tan(2x+1)$

$\frac{dy}{dx} = -2\sec^2(2x+1)$

(e) $\frac{1}{3}\sec(3x-2)$

$y = \frac{1}{3}\sec(3x-2)$

$\frac{dy}{dx} = \frac{1}{3}\cdot 3\sec(3x-2)\tan(3x-2)$

$\frac{dy}{dx} = \sec(3x-2)\tan(3x-2)$

(f) $\tan\sqrt{x}$

$y = \tan\sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{-1}{2}}\sec^2\sqrt{x}$

$\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Example II

Differentiate the following:

(a) $x\tan x$

(b) $\sec x\tan x$

(c) $x^2\cot x$

(d) $3x\,\text{cosec}\,x$

(e) $\text{cosec}\,x\cot x$

(f) $\frac{\tan x}{x}$

Solutions

(a) $x\tan x$

$y = x\tan x$

$\frac{dy}{dx} = x\sec^2 x + (\tan x).1$

$\frac{dy}{dx} = x\sec^2 x + \tan x$

(b) $\sec x\tan x$

$y = \sec x\tan x$

$\frac{dy}{dx} = \sec x\sec^2 x + \tan x\cdot(\sec x\tan x)$

$\frac{dy}{dx} = \sec^3 x + \tan^2 x\sec x.$

(d) $3x\,\text{cosec}\,x$

$y = 3x\,\text{cosec}\,x$

$\frac{dy}{dx} = 3x(-\text{cosec}\,x\cot x) + \text{cosec}\cdot 3$

$\frac{dy}{dx} = -3x\,\text{cosec}\,x\cot x + 3\text{cosec}\,x$

(e) $\text{cosec}\,x\cot x$

$y = \text{cosec}\,x\cot x$

$\frac{dy}{dx} = \text{cosec}\,x\cdot -\text{cosec}^2 x + (\cot x)(-\cot x\,\text{cosec}\,x)$

$\frac{dy}{dx} = \text{cosec}^3 x - \cot^2 x\,\text{cosec}\,x$

Example III

Differentiate the following

(a) $\tan^2 x$

(b) $\sec^2 x$

(c) $3\text{cosec}^2 x$

(d) $-\tan^2 2x$

(e) $\frac{1}{2}\cot^2 3x$

(f) $\sqrt{\tan x}$

(g) $-2\text{cosec}^4 x$

Solution

(a) $\tan^2 x$

$y = \tan^2 x$

$$\frac{dy}{dx} = 2\tan x(\sec^2 x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(b) $\sec^2 x$

$$y = \sec^2 x$$
$$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$$
$$\frac{dy}{dx} = 2\sec^2 x \tan x$$

(c) $3\text{cosec}^2 x$

$$y = 3\text{cosec}^2 x$$
$$\frac{dy}{dx} = 3 \times 2\text{cosec}\, x(-\text{cosec}\, x \cot x)$$
$$\frac{dy}{dx} = -6\text{cosec}^2 x\cot x$$

(d) $-\tan^2 2x$

$$y = -\tan^2 2x$$
$$\frac{dy}{dx} = -2(\tan 2x)(2\sec^2 2x)$$
$$\frac{dy}{dx} = -4\sec^2 2x \tan 2x$$

(e) $\frac{1}{2}\cot^2 3x$

$$y = \frac{1}{2}\cot^2 3x$$
$$\frac{dy}{dx} = \frac{1}{2} \times 2 \cot 3x(-3\text{cosec}^2 3x)$$
$$= -3\text{cosec}^2 3x\cot 3x$$

(f) $\sqrt{\tan x}$

$$\frac{dy}{dx} = \frac{1}{2}(\tan x)^{\frac{-1}{2}} \cdot \sec^2 x$$
$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

(g) $-2\text{cosec}^4 x$

$$y = -2\text{cosec}^4 x$$
$$\frac{dy}{dx} = -8\text{cosec}^3 x(-\text{cosec}\, x \cot x)$$
$$\frac{dy}{dx} = 8\text{cosec}^4 x \cot x$$

Integration of Trigonometric functions

Integration is the process of obtaining a function from its derivative

Note: $\int \cos ax \, dx = \frac{1}{a}\sin(ax) + c$

$$\int \sin ax \, dx = \frac{-1}{a}\cos(ax) + c$$

Example I

Integrate the following

(a) $\cos 3x$
(b) $\sin 3x$
(c) $\cos(3x - 1)$
(d) $\sin(2x + 1)$
(e) $6 \cos 4x$

Solution

(a) $\cos 3x$

$$y = \cos 3x$$
$$\int y\, dx = \int \cos 3x \, dx$$
$$= \frac{1}{3}\sin 3x + c$$
$$\int \cos 3x \, dx = \frac{1}{3}\sin 3x + c$$

(b) $\int \sin 3x \, dx = \frac{1}{3}\sin 3x + c$

$$= \frac{-1}{3}\cos 3x + c$$

(c) $\int \cos(3x - 1)\, dx = \frac{1}{3}\sin(3x - 1) + c$

(d) $\int \sin(2x + 1)\, dx = \frac{-1}{2}\cos(2x + 1) + c$

(e) $\int 6\cos 4x \, dx = 6\int \cos 4x \, dx$

$$= 6\left[\frac{1}{4}\sin 4x\right] + c$$
$$= \frac{3}{2}\sin 4x + c$$

Example

Integrate the following

(a) $\sec^2 2x$
(b) $3\sec x \tan x$
(c) $-\text{cosec}^2 \frac{1}{2} x$
(d) $\frac{1}{3}\text{cosec}\, 3x \cot 3x$
(e) $2\sec^2 x \tan x$

(f) $\frac{\sin x}{\cos^2 x}$

(g) $\frac{1}{\sin^2 2x}$

(h) $\frac{\cos 2x}{\sin^2 2x}$

Solution

Note: $\frac{d}{dx}(\tan x) = \sec^2 x$

$\Rightarrow \int \sec^2 x\, dx = \tan x + c$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\Rightarrow \int \sec x \tan x\, dx = \sec x + c$

$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

$\Rightarrow \int \text{cosec}^2 x\, dx = -(\cot x) + c$

$\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

$\Rightarrow \int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$

(a) $\int \sec^2 2x\, dx$

Let $u = 2x$

$du = 2dx$

$dx = \frac{du}{2}$

$$\int \sec^2 2x\, dx = \int \sec^2 u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int \sec^2 u\, du$$

$$= \frac{1}{2}\tan u + c$$

$$= \frac{1}{2}\tan(2x) + c$$

(b) $\int 3\sec x \tan x\, dx$

$= 3\int \sec x \tan x\, dx$

$= 3\sec x + c$

(c) $\int {}^{-}\text{cosec}^2 \tfrac{1}{2}x\, dx$

Let $u = \frac{1}{2}x$

$\frac{du}{dx} = \frac{1}{2}$

$dx = 2\, du$

$$\int -\text{cosec}^2 \tfrac{1}{2}x\, dx = \int -\text{cosec}^2 u(2du)$$

$$= 2\int -\text{cosec}^2 u$$

$$= 2[\cot u] + c$$

$$= 2\cot \tfrac{1}{2}x + c$$

(d) $\int \tfrac{1}{3}\text{cosec}\, 3x \cot 3x\, dx$

Let $u = 3x$

$du = 3\, dx$

$dx = \frac{du}{3}$

$$\int \tfrac{1}{3}\text{cosec}\, 3x \cot 3x\, dx = \frac{1}{3}\int \text{cosec}\, 3x \cot 3x\, dx$$

$$= \frac{1}{3}\int \text{cosec}\, u \cot u \cdot \frac{du}{3}$$

$$= \frac{1}{9}\int \text{cosec}\, u \cot u\, du$$

$$= \frac{1}{9}(-\text{cosec}\, u) + c$$

$$= \frac{-1}{9}\text{cosec}\, 3x + c$$

(e) $\int 2\sec^2 x \tan x\, dx$

Consider $\frac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x)$

$= 2\sec^2 x \tan x$

$\Rightarrow \int 2\sec^2 x \tan x\, dx = \sec^2 x + c$

(f) $\int \frac{\sin x}{\cos^2 x} dx$

$$= \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$$

$$= \int \tan x \sec x\, dx$$

$= \sec x + c$

(g) $\frac{1}{\sin^2 2x} = \int \text{cosec}^2 2x\, dx$

Let $u = 2x$

$du = 2\, dx$

$dx = \frac{du}{2}$

$$\int \text{cosec}^2 2x\, dx = \int \text{cosec}^2 u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int \text{cosec}^2 u$$
$$= \frac{-1}{2}\cot u + c$$
$$= \frac{-1}{2}\cot 2x + c$$

(h) $\dfrac{\cos 2x}{\sin^2 2x}$

$$= \int \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\sin 2x}\, dx$$
$$= \int \cot 2x \,\text{cosec}\, 2x\, dx$$

Let $u = 2x$

$du = 2\, dx$

$$dx = \frac{du}{2}$$

$$\int \cot u \,\text{cosec}\, u \cdot \frac{du}{2}$$
$$= \frac{1}{2}\int \text{cosec}\, u \cot u\, du$$
$$= \frac{1}{2}(-\text{cosec}\, u) + c$$
$$= \frac{-1}{2}\text{cosec}\, 2x + c$$

Example III

Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\, dx$

(c) $\int_0^{\pi} \sin^2 x\, dx$

Solution

(a) $\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

$$= \left[\frac{-1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{-1}{2}\cos 2\left(\tfrac{\pi}{2}\right) - \frac{-1}{2}\cos 0$$
$$= \frac{-1}{2}(-1) + \frac{1}{2}$$
$$= 1$$

(b) $\int_{\frac{-\pi}{3}}^{\frac{\pi}{6}} \sec^2 x\, dx$

$$= [\tan x]_{\frac{-\pi}{3}}^{\frac{\pi}{6}}$$
$$= \tan\left(\tfrac{\pi}{6}\right) - \tan\left(\tfrac{-\pi}{3}\right)$$
$$= \frac{1}{\sqrt{3}} - \left(-\sqrt{3}\right)$$
$$= \frac{1}{\sqrt{3}} + \sqrt{3}$$
$$= \frac{\sqrt{3}}{3} + \sqrt{3} = \frac{4\sqrt{3}}{3}$$

(c) $\int_0^{\pi} \sin^2 x\, dx$

From $\cos 2x = 1 - 2\sin^2 x$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1}{2}(1 - \cos 2x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$= \frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - (0 - 0)\right]$$
$$= \frac{1}{2}[\pi]$$
$$= \frac{1}{2}\pi$$

Example

A particle moves in a straight line such that its velocity in m/s after passing through a fixed point O is $3\cos t - 2\sin t$. Find:

(a) Its distance from O after $\frac{1}{2}\pi$ s

(b) Its acceleration after π s

(c) The time when its velocity is first zero.

Solution

$V = 3\cos t - 2\sin t$

$$\frac{dS}{dt} = 3\cos t - 2\sin t$$

$dS = (3\cos t - 2\sin t)\, dt$

$S = 3\sin t + 2\cos t + c$

When $t = 0$, $S = 0$

$0 = 3\sin(0) + 2\cos(0) + c$

$-2 = c$

$\Rightarrow S = 3\sin t + 2\cos t - 2.$

When $t = \frac{1}{2}\pi$,

$S = 3\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} - 2$

$S = 3 - 2$

$S = 1$ m

$V = 3\cos t - 2\sin t$

$a = \frac{dV}{dt} = -3\sin t - 2\cos t$

$a = \left.\frac{dV}{dt}\right|_{\pi} = -3\sin \pi - 2\cos \pi$

$= 2$ m/s^2

$\Rightarrow a = 2$ m/s^2

$V = 3\cos t - 2\sin t$

$3\cos t - 2\sin t = 0.$

$R = \cos(t + \alpha) = 0$

$R = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\sqrt{13}\cos(t + \alpha) = 0$

$\sqrt{13}\cos(t + 33.7) = 0$

$\cos(t + 33.7) = 0$

$t + 33.7 = \cos^{-1}0$

$t + 33.7 = 90$

t = 56.3

$t = \frac{56.3\pi}{180}$

$t = 0.983$ s

Revision Exercise

1. Solve the following for all values of x from 0° to 360°.

(a) $\sin x = \frac{1}{2}$

(b) $\cos x = \frac{-1}{2}$

(c) $\tan x = 1$

(d) $\tan x = -1$

(e) $\sin x = \frac{-\sqrt{3}}{2}$

(f) $\cos x = \frac{1}{\sqrt{2}}$

2. Solve the following equations for values of x from -180° to 180°

(a) $\sin x = \frac{-1}{2}$

(b) $\cos x = \frac{1}{2}$

(c) $\sin x = \frac{\sqrt{3}}{2}$

(d) $\tan x = \sqrt{3}$

(e) $\cos x = \frac{-1}{\sqrt{2}}$

(f) $\cos x = \frac{-\sqrt{3}}{2}$

3. Solve the following equations for all values of x from 0° to 360°

(a) $\sin x = \frac{-1}{2}$

(b) $\cos x = -0.7$

(c) $\tan x$ -0.75

(d) $\cos^2 x = \frac{1}{4}$

(e) $\sin x = 2\cos x$

(f) $2\sin x - 3\cos x = 0$

(g) $\sin 2x = \frac{-\sqrt{3}}{2}$

(h) $\cos 2x = \frac{1}{2}$

(i) $\sin(x + 20) = \frac{-\sqrt{3}}{2}$

(j) $\tan(x - 30) = 1$

(k) $3(\cos x - 1) = -1$

(l) $\sin x\,(1 - 2\cos x) = 0$

(m) $\cos x(2\sin x + \cos x) = 0$

(n) $2\sin x\cos x + \sin x = 0$

(o) $4\sin x\cos x = 3\cos x$

(p) $4\cos^2 x + \cos x = 0$

(q) $\tan x = 4\sin x$

(r) $(2\sin x - 1)(\sin x + 1) = 0$

(s) $2\sin^2 x - \sin x - 1 = 0$

(t) $2\tan^2 x - \tan x - 6 = 0$

(u) $2\tan x - \frac{1}{\tan x} = 1$

Solve the following equations for all values of x from -180° to 180°

1. $\cos^2 x = \frac{3}{4}$
2. $\sin 2x = 2\cos 2x$
3. $\cos(x - 20) = \frac{-1}{\sqrt{2}}$
4. $\cos x(\sin x - 1) = 0$
5. $3\sin^2 x = 2\sin x\cos x$
6. $2\cos^2 x - 5\cos x + 2 = 0$

7. Factorise the expression $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2$. Hence solve $6\sin\theta\cos\theta + 3\cos\theta + 4\sin\theta + 2 = 0$ for $-180° \leq 180°$
8. Factorise the equation $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta - 2$. Hence solve $3\sin\theta\cos\theta - 3\sin\theta + 2\cos\theta = 2$.
9. Without using tables or calculator, find the values of:
   (a) sec 45°
   (b) cot 45°
   (c) cosec 30
   (d) sec 60°
   (e) cosec 135°
   (f) sec 120°
   (g) cosec 330°
   (h) sec 240°
   (i) cot -135°
   (j) sec -60°
   (k) sec(-120°)
   (l) cosec 315°
10. Simplify the following expression:
   (a) $\sqrt{(1-\sin A)(1+\sin A)}$
   (b) $\operatorname{cosec}\theta\tan\theta$
   (c) $\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$
   (d) $\cot\theta\sqrt{1-\cos^2\theta}$
11. Prove the following identities
   (a) $\sin\theta\tan\theta + \cos\theta = \sec\theta$
   (b) $\operatorname{cosec}\theta - \sin\theta = \cot\theta\cos\theta$
   (c) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2\theta = 2$
   (d) $(\sin\theta + \operatorname{cosec}\theta)^2 = \sin^2\theta + \cot^2\theta + 3\theta$
   (e) $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$
   (f) $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$
   (g) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$
   (h) $\frac{\operatorname{cosec}\theta}{\cos\theta + \tan\theta} = \cot\theta$
   (i) $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$
   (j) $\frac{\sin\theta}{1-\cos\theta} + \frac{\sin\theta}{1+\cos\theta} = 2\operatorname{cosec}\theta$
   (k) $\cos^4 x - \sin^4 x = \cos^2 x$
   (l) $\cos A + \cos(B + C) = 0$
   (m) $\frac{\cos^2\theta}{1+\cot^2\theta} = 2\cos\theta$

12. Prove the following identities:
   (a) $2\operatorname{cosec} 2\theta = \operatorname{cosec}\theta\sec\theta$
   (b) $\tan A + \cot A = 2\text{coesec } 2A$
   (c) $\frac{1+\tan^2 A}{2-\tan^2 A} = \sec 2A$
   (d) $\cot 2A = \operatorname{cosec} 2A - \tan A$
   (e) $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$
   (f) $\tan\theta - \cot\theta = -2\cot 2\theta$

13. Prove the following identities:
   (a) $\frac{1+\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$
   (b) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$
   (c) $\frac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

14. Eliminate θ from each of the following pairs of relationships
   (a) $x = \sin\theta$, $y = \cos\theta$
   (b) $x = 3\sin\theta$, $y = \operatorname{cosec}\theta$
   (c) $5x = \sin\theta$, $y = 2\cos\theta$
   (d) $x = 3 + \sin\theta$, $y = \cos\theta$
   (e) $x = 2 + \sin$, $\cos\theta = 1 + y$.
15. Solve the following equations for all values of θ from -180° to 180°.
   (a) $4 - \sin\theta = 4\cos^2\theta$
   (b) $\sin^2\theta + \cos\theta + 1 = 0$
   (c) $5 - 5\cos\theta = 3\sin^2\theta$
   (d) $8\tan\theta = 3\cos\theta$
   (e) $\sin^2\theta + 5\cos^2\theta = 0$
   (f) $1 - \cos^2\theta = -2\sin\theta\cos\theta$
16. Solve the following equations from 0° to 360°
   (a) $\sec\theta = 2$
   (b) $\cot 2\theta = \frac{-2}{5}$
   (c) $3\cot\theta = \tan\theta$
   (d) $2\sin\theta = -3\cot\theta$
   (e) $2\sec^2\theta - 3 + \tan\theta = 0$
17. If $A + B + C = 180°$, prove that
$$\cos 2A + \cos 2B + \cos 2C = 1 - \cos A\cos B\cos C$$
18. Prove that sin 3A = 4sinAsin(60 + A)sin(60 – A)
19. Show that in a triangle ABC, if $2S = a + b + c$, then
$$1 - \tan\tfrac{A}{2} + \tan\tfrac{B}{2} = \frac{C}{A}$$
20. Prove that in any triangle ABC,
$$(a + b + c)(\tan\tfrac{A}{2} + \tan\tfrac{B}{2}) = 2c\cot\tfrac{C}{2}.$$
21. Prove that in any triangle
$$ABC,\ \frac{a+b-c}{a+b+c} = \tan\tfrac{A}{2}\tan\tfrac{B}{2}$$
22. From a point A, a light wind due to north of A has an elevation α from a point B, due west of A. The angle

of elevation is β. Prove that the angle of elevation from the midpoint of *AB*. is $\tan^{-1}\left(\frac{2}{\sqrt{3\cot^2\alpha+\cot^2\beta}}\right)$

23. Solve: $4\cos\alpha - 3\sin\alpha = 2$
24. Solve the equation $15\cos2\theta + 20\sin2\theta + 7 = 0$
25. Find all the possible values of x that satisfy $\tan^{-1}3x + \tan^{-1}x = \frac{\pi}{4}$
26. Prove that $\frac{\sin8\theta\cos\theta - \sin6\theta\cos3\theta}{\cos2\theta\cos\theta - \sin3\theta\sin4\theta} = \tan2\theta$
27. Solve the equation $2\cos^2(x-\frac{\pi}{2}) - 3\cos(x-\frac{\pi}{2}) = 0$ for $0 \le x \le 2\pi$.
28. Solve $\cos^4x + \sin^4x = \frac{7}{8}$ for $0 \le x \le \frac{\pi}{2}$.
29. Find the value of x for $3\cos^2x - 8\cos x + 4 = 0$
30. Show that $\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2 = \frac{1-\cos\theta}{1+\cos\theta}$
31. Prove that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$
32. Solve the equation $\cos x - \cos 4x = \cos 2x - \cos 3x$ for $-\pi \le x \le \pi$.
33. Given that $y = 4\cos x - 6\sin x$. Express y in the form $R\cos(x + \alpha)$, where R is a constant. Find the maximum and minimum value of y.
34. Express (45° + x) in terms of tan x. Hence or otherwise express tam 75° in the form $a + b\sqrt{3}$.
35. Given $\sin x = \frac{-4}{5}$, where $180° \le x \le 270$, find without using tables or calculator the value of tan $3x$.
36. Show that:

    (a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

    (b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

    (c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

37. Solve the equation

    (a) $\tan^{-1}(2x + 1) + \tan^{-1}(2x - 1) = \tan^{-1}2$

    (b) $\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = 32$

    (c) $\cos^{-1}x + \cos^{-1}x\sqrt{8} = \frac{\pi}{2}$

    (d) $2\sin\frac{x}{2} + \sin^{-1}x\sqrt{2} = \frac{\pi}{2}$

38. Without using tables or calculator, evaluate $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{5}$