

COMPLEX NUMBERS

A complex number is represented by an expression of the form $a + bi$ where a and b are real numbers and i is a symbol with a property $i^2 = -1$.

$i = \sqrt{-1}$ was introduced by a Swiss mathematician Euler. Traditionally the letters Z and W are used to stand for complex numbers.

Given a complex numbers $z = a + bi$.

The real part of a complex number z is $Re(z) = a$ and the imaginary part of z is $Im(z) = b$.

Both $Re(z)$ and $Im(z)$ real numbers.

Thus the real part of $Z = 4 - 3i$ is $Re(w) = 4$ and imaginary part of Z is $Im(Z) = -3$

By identifying the real number a with a complex number $a + 0i$ we consider \mathbb{R} (real numbers) to be subset of \mathbb{C} (complex numbers).

Consider the equation $x^2 + 9 = 0$, this can be written as $x^2 = -9$ and we can see that the equation has no real roots since we cannot find the real root of a negative number, But with $i^2 = -1$ (Euler) we are able to find the square root of complex numbers.

$$\begin{aligned}x^2 &= -9 \\x^2 &= 9i^2 \\ \sqrt{x^2} &= \sqrt{9i^2} \\ x &= \pm 3i \\ x &= 3i \quad x = -3i\end{aligned}$$

Example

Solve the following equations

(a) $4x^2 + 49 = 0$

(b) $x^2 + 2x + 6 = 0$

Solution

$$4x^2 + 49 = 0$$

$$4x^2 = -49$$

$$x^2 = -\frac{49}{4}$$

$$x^2 = \frac{49}{4}i^2$$

$$\begin{aligned}x &= \sqrt{\frac{49}{4}i^2} \\ x &= \pm \frac{7}{2}i\end{aligned}$$

(b) $x^2 + 2x + 6 = 0$

$$\text{From, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(-2 \pm \sqrt{(2)^2 - 4(1)(6)})}{2 \times 1}$$

$$x = \frac{-2 \pm \sqrt{-20}}{2}$$

$$x = \frac{-2 \pm \sqrt{4i^2 \times 5}}{2}$$

$$x = \frac{-2 \pm 2i\sqrt{5}}{2}$$

$$x = -1 + i\sqrt{5}$$

$$x = -1 - i\sqrt{5}$$

With this new concept we are in position to find the roots of any quadratic equation.

When the imaginary part of a complex number is zero, the complex number becomes a real number. Thus, all real numbers are complex numbers.

Definition

Given a complex number $z = x + iy$, the complex conjugate of Z denoted by \bar{z} or z^* is a complex number given by $\bar{z} = x - iy$. Therefore if $z = 4 + 3i$, $w = -2 + 4i$

Then $\bar{z} = 4 - 3i$, $\bar{w} = -2 - 4i$

Algebra of complex numbers

1. Addition

Given that two complex numbers

$$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2. \text{ Then}$$

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$$

$$= x_1 + x_2 + i(y_1 + y_2)$$

Therefore if $z_1 = 3 + 5i$ and $z_2 = 2 - 7i$

$$z_1 + z_2 = 3 + 5i + 2 - 7i$$

$$= (3 + 2) + 5i - 7i$$

$$= 5 - 2i$$

Example

1. Subtraction:

$$\begin{aligned}
 z_1 &= x_1 + iy_1 \\
 z_2 &= x_2 + iy_2 \\
 z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\
 &= x_1 - x_2 + iy_1 - iy_2 \\
 &= (x_1 - x_2) + i(y_1 - y_2) \\
 z_1 &= 4 - 3i \\
 z_2 &= 6 - 14i
 \end{aligned}$$

Find $(z_1 - z_2)$

$$\begin{aligned}
 (z_1 - z_2) &= (4 - 3i) - (6 - 14i) \\
 &= 4 - 6 - 3i + 14i \\
 &= -2 + 11i
 \end{aligned}$$

2. Multiplication

$$\begin{aligned}
 z_1 &= x_1 + iy_1, \quad z_2 = x_2 + iy_2 \\
 z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
 &= x_1 x_2 + x_1 y_2 i + y_1 x_2 i + i^2 y_1 y_2 \\
 &= x_1 x_2 - y_1 y_2 + (y_1 x_2 + x_1 y_2) i
 \end{aligned}$$

Example

$$z_1 = 3 + 5i, \quad z_2 = 2 - 7i$$

Find $z_1 z_2$

Solution

$$\begin{aligned}
 \text{Find } z_1 z_2 &= (3 + 5i)(2 - 7i) \\
 &= 3(2 - 7i) + 5i(2 - 7i) \\
 &= 6 - 21i + 10i - 35i^2 \\
 &= 6 + 35 + (10 - 21)i \\
 &= 41 - 11i
 \end{aligned}$$

3. Division

$$\begin{aligned}
 z_1 &= x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \\
 \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\
 \frac{z_1}{z_2} &= \frac{x_1 + iy_1(x_2 - iy_2)}{x_2 + iy_2(x_2 - iy_2)} \\
 \frac{z_1}{z_2} &= \frac{x_1 x_2 - x_1 y_2 i + x_2 y_1 i - i^2 y_1 y_2}{(x_2)^2 - i^2 y_2^2} \\
 &= \frac{x_1 x_2 + y_1 y_2 + (x_2 y_1 - x_1 y_2) i}{x_2^2 + y_2^2} \\
 &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{(x_2 y_1 - x_1 y_2) i}{x_2^2 + y_2^2}
 \end{aligned}$$

Example I

$$\text{Simplify } z = \frac{2+6i}{3-i}$$

$$\begin{aligned}
 z &= \frac{2 + 6i}{3 - i} = \frac{2 + 6i(3 + i)}{(3 - i)(3 + i)} \\
 &= \frac{2(3 + i) + 6i(3 + i)}{3^2 - i^2} \\
 &= \frac{6 + 2i + 18i + 6i^2}{10} \\
 &= \frac{6 + 20i - 6}{10} \\
 &= \frac{0 + 20i}{10} \\
 &= 2i
 \end{aligned}$$

Example II

Express $\frac{-1+2i}{1+3i}$ in the form $a + bi$

Solution

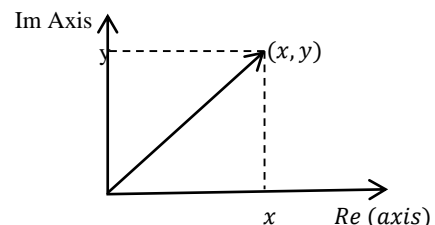
$$\begin{aligned}
 \frac{-1 + 2i}{1 + 3i} &= \frac{-1 + 2i(1 - 3i)}{1 + 3i(1 - 3i)} \\
 &= \frac{-1 + 3i + 2i - 6i^2}{(1)^2 - (3i)^2} \\
 &= \frac{5i + 5}{10} \\
 &= \frac{5 + 5i}{10} \\
 &= \frac{1}{2} + \frac{1}{2}i
 \end{aligned}$$

The Argand Diagram

Complex numbers can be represented graphically on a graph of Real (Re) and Imaginary (Im) axes called a **complex plane**. The complex plane is similar to the Cartesian plane where the imaginary axis corresponds to the y-axis and the real axis corresponds to the x-axis. The diagram representing the complex number in complex plane is called an **argand diagram** named after JR argand 1806.

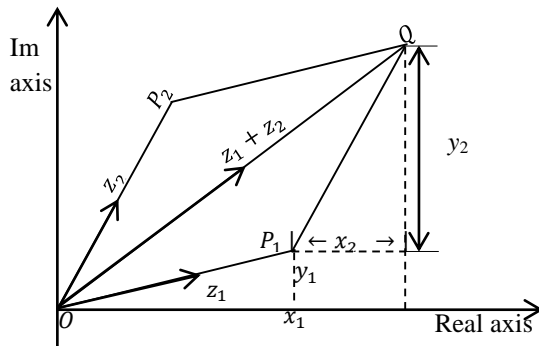
On the argand diagram a complex number is represented by a line with an arrow on the head to show direction

If $z = x + iy$ we can represent z on argand diagram as shown below.



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

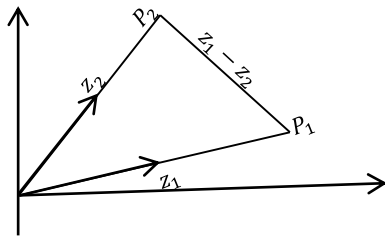


z_1, z_2 and $z_1 + z_2$ is represented by vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ and \overrightarrow{OQ} respectively. The diagram shows that $\overrightarrow{P_1Q}$ is equal $\overrightarrow{OP_2}$ in magnitude and direction

$$\overrightarrow{OQ} = \overrightarrow{OP_1} + \overrightarrow{P_1Q} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$$

Thus the sum of two complex numbers z_1 and z_2 is represented in the argand diagram by the sum of the corresponding vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$

Representing $z_1 - z_2$ on the argand diagram.



$$(z_1 - z_2) = OP_1 - OP_2 = \overrightarrow{P_1P_2}$$

Since $\overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{P_1P_2}$

$z_1 - z_2$ can be represented by P_1P_2

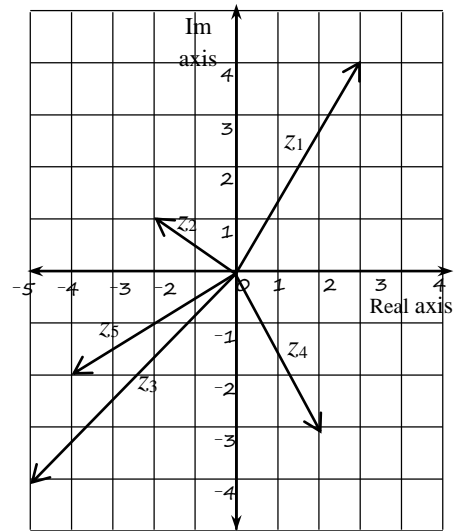
Example

Represent the following complex numbers on the argand diagram.

$$z_1 = 3 + 4i, \quad z_2 = -2 + i, \quad z_3 = -5 - 4i,$$

$$z_4 = 2 - 3i, \quad z_5 = -4 - 2i,$$

Solution



Modulus of a complex number

Given a complex number $z = x + iy$, the magnitude or length of z is denoted by $|z|$ is defined by

$$|z| = \sqrt{x^2 + y^2}$$

Example I

Given $z = 1 + \sqrt{3}i$ find $|z|$

Solution

$$z = 1 + (\sqrt{3})i$$

$$\begin{aligned} |z| &= \sqrt{(1)^2 + (\sqrt{3})^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Example II

Find $|z|$ if $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution

$$\begin{aligned} z &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ |z| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\frac{1}{4} + \frac{3}{4}} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\Rightarrow |z| = 1$$

Example III

$z = -3 + 4i$ find $|z|$

Solution

$$\begin{aligned}
 z &= -3 + 4i \\
 |z| &= \sqrt{(-3)^2 + (4)^2} \\
 &= \sqrt{9 + 16} \\
 &= \sqrt{25} \\
 &= 5
 \end{aligned}$$

Properties of modulus

If z_1 and z_2 are complex numbers then

$$(i) |z_1 z_2| = |z_1| |z_2|$$

$$(ii) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

Example I

$$z_1 = 5 - 12i \text{ and } z_2 = 3 - 4i$$

Find $|z_1 z_2|$ and $\left| \frac{z_1}{z_2} \right|$

Solution

$$z_1 = 5 - 12i, \quad z_2 = 3 - 4i$$

$$\begin{aligned}
 |z_1 z_2| &= |z_1| |z_2| \\
 &\Rightarrow |(5 - 12i)(3 - 4i)| \\
 &= |5 - 12i| |3 - 4i| \\
 &= \sqrt{(5)^2 + (-12)^2} \sqrt{(3)^2 + (-4)^2} \\
 &= \sqrt{169} \times \sqrt{25} \\
 &= 13 \times 5 \\
 &= 65
 \end{aligned}$$

Alternatively

$$\begin{aligned}
 z_1 z_2 &= (5 - 12i)(3 - 4i) \\
 &= 15 - 20i - 36i + 48i^2 \\
 &= 15 - 48 - 56i \\
 &= -33 - 56i \\
 |z_1 z_2| &= \sqrt{(-33)^2 + (-56)^2} \\
 &= 65 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 z_1 &= 5 - 12i, \quad z_2 = 3 - 4i \\
 \left| \frac{z_1}{z_2} \right| &= \frac{|z_1|}{|z_2|} = \frac{\sqrt{(5)^2 + (-12)^2}}{\sqrt{(3)^2 + (-4)^2}} \\
 &= \frac{13}{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{Alternatively, } \frac{z_1}{z_2} &= \frac{5 - 12i}{3 - 4i} \\
 &= \frac{(5 - 12i)(3 + 4i)}{(3 - 4i)(3 + 4i)} \\
 \frac{z_1}{z_2} &= \frac{15 + 20i - 36i - 48i^2}{(3)^2 - (4i)^2}
 \end{aligned}$$

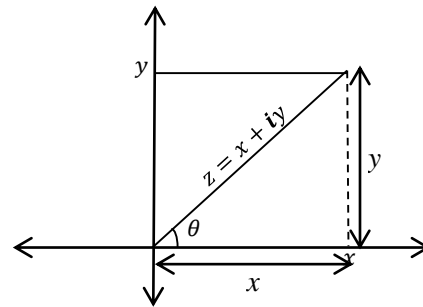
$$\begin{aligned}
 &= \frac{63 - 16i}{9 + 16} \\
 &= \frac{63}{25} - \frac{16}{25}i
 \end{aligned}$$

$$\left| \frac{z_1}{z_2} \right| = \sqrt{\left(\frac{63}{25}\right)^2 + \left(-\frac{16}{25}\right)^2}$$

$$\begin{aligned}
 \left| \frac{z_1}{z_2} \right| &= \sqrt{\frac{(63)^2 + (16)^2}{25^2}} \\
 &= \frac{65}{25} = \frac{13}{5}
 \end{aligned}$$

Argument of a complex number Z (arg Z)

The argument of a complex number z is defined to be the angle (θ) which the complex number z makes with the positive x -axis.



From the diagram above,

$$\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

Note: For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360° .

To avoid confusion we usually work with the value of θ for which $-\pi < \theta < \pi$ or $-180 < \theta < 180$. This is called the principle argument of z denoted by **arg z**.

$$\begin{aligned}
 \text{In practice the formula } \tan \theta &= \frac{y}{x} \\
 \theta &= \tan^{-1} \left(\frac{y}{x} \right)
 \end{aligned}$$

Which is often used to find the principal argument of a complex number z , despite the fact that it tends to two possible values for θ in the permitted range. The formula is necessary but not sufficient to help us

obtain the arg z . The correct value of $\arg z$ is chosen with the aid of a sketch.

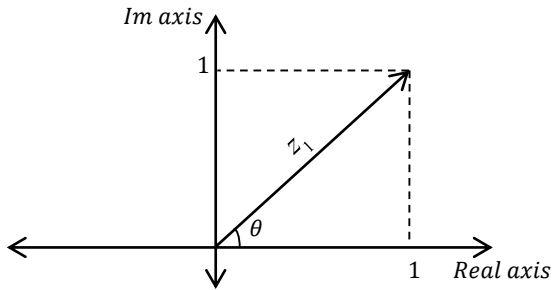
Example

Find the principal argument of the following complex number

- (a) $1 + i$ (b) $-1 - i\sqrt{3}$ (c) -5
 (d) $-\sqrt{3} + i$ (e) $\sqrt{3} - i$

Solution

Consider $z_1 = 1 + i$



$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$$

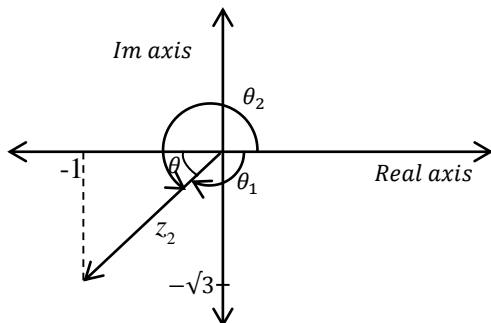
Since $180^\circ = \pi$ radians

$$\theta = \frac{45\pi}{180} = \frac{\pi}{4}$$

$$\Rightarrow \arg z_1 = 45^\circ$$

$$\arg z_1 = \frac{45\pi}{180} = \frac{\pi}{4}$$

(b) Let $z_2 = -1 - i\sqrt{3}$



$$\tan \theta = \frac{\sqrt{3}}{1}$$

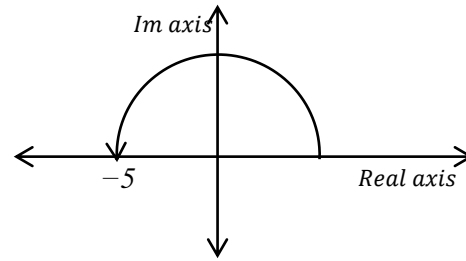
$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\theta = 60^\circ$$

$$\arg z_2 = \theta_1$$

$$\Rightarrow \arg z_2 = -120^\circ$$

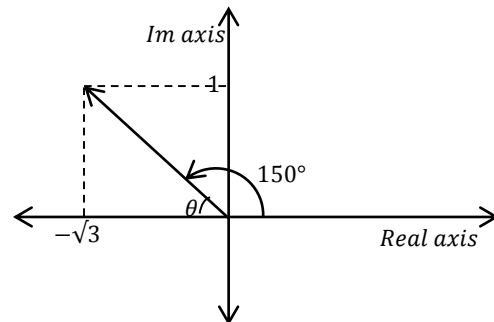
OR $\arg z_2 = -\frac{2}{3}\pi$



$$z_3 = -5 + 0i$$

$$\arg z_3 = 180^\circ \text{ or } \arg z_3 = \pi$$

(d) Let $z_4 = -\sqrt{3} + i$



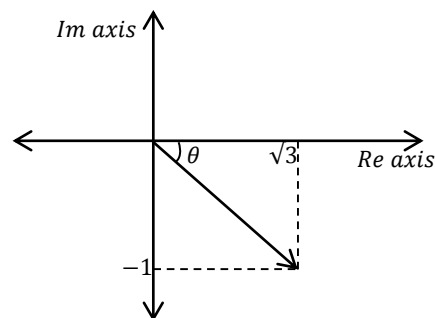
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$z_4 = -\sqrt{3} + i$$

$$\arg z_4 = 150^\circ, \text{ from the sketch above}$$

(e) $\sqrt{3} - i$

$$z_5 = \sqrt{3} - i$$



$$\tan \theta = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$\arg z_5 = -30^\circ \text{ from the above diagram}$$

Properties of Arguments

Given the two complex numbers z_1 and z_2 then

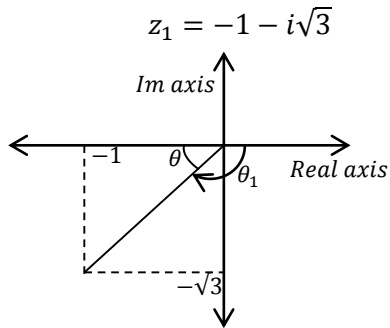
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Example I

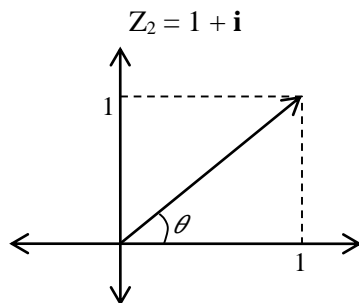
Given that $z_1 = -1 - i\sqrt{3}$ and $z_2 = 1 + i$. Find the $\arg(z_1 z_2)$ and $\arg\left(\frac{z_1}{z_2}\right)$

Solution



$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$$

$$\arg z_1 = \theta_1 = -120^\circ$$



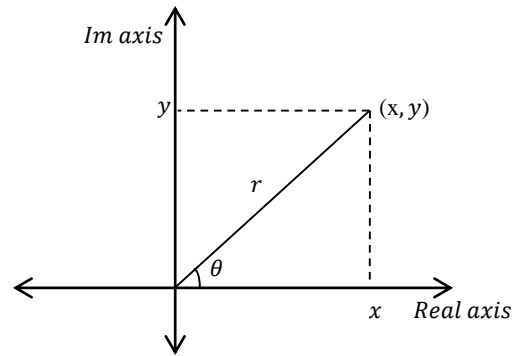
$$\arg z_2 = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$$

$$\begin{aligned}\arg(z_1 z_2) &= \arg z_1 + \arg z_2 \\ &= -120 + 45^\circ \\ &= -75^\circ\end{aligned}$$

$$\begin{aligned}\arg\left(\frac{z_1}{z_2}\right) &= \arg z_1 - \arg z_2 \\ &= -120 - 45 \\ &= -165\end{aligned}$$

Modulus–argument form of a complex number

(Polar form of a complex number)



Consider a complex number $z = x + iy$ making an angle θ with the positive x – axis

$$\arg z = \theta$$

From the diagram above $\cos \theta = \frac{x}{r}$ $\sin \theta = \frac{y}{r}$

$$x = r \cos \theta \quad r \sin \theta = y$$

$$z = x + iy$$

$$z = r \cos \theta + ir \sin \theta$$

$$z = r (\cos \theta + i \sin \theta)$$

(modulus argument form a complex number)

Where $r = |z| = \sqrt{x^2 + y^2}$

Example

Express the following complex numbers in modulus –argument

- $5 + 5i\sqrt{3}$
- $\sqrt{2} + i$
- $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$
- $-3\sqrt{2} + 3\sqrt{2}i$
- $-5i$
- $-5 - 12i$

Solutions

$$z_1 = 5 + 5i\sqrt{3}$$

$$\begin{aligned}r &= \sqrt{(5)^2 + (5\sqrt{3})^2} \\ &= \sqrt{25 + 75} \\ &= 10\end{aligned}$$

$$\arg z_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = 60^\circ$$

$$z_1 = 5 + 5i\sqrt{3} = 10(\cos 60 + i \sin 60)$$

(b) $z_2 = \sqrt{2} + i$

$$\begin{aligned}|z_2| = r &= \sqrt{(\sqrt{2})^2 + (1)^2} \\ &= \sqrt{3}\end{aligned}$$

$$\arg z_2 = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ$$

$$z_2 = r(\cos \theta + i \sin \theta)$$

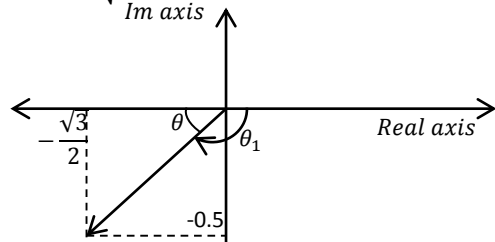
$$\sqrt{3}[\cos 35.3 + i \sin 35.3]$$

(c) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$$z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$|z_3| = \sqrt{\left(\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right)}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$



$$\theta = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = 30^\circ$$

$$z_3 = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$$

$$\arg z_3 = -150^\circ$$

$$z_3 = r(\cos \theta + i \sin \theta)$$

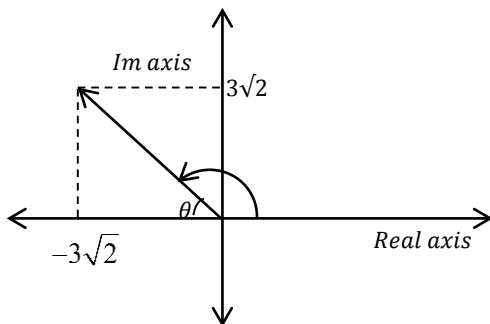
$$z_3 = 1(\cos -150 + i \sin -150)$$

(d) $z_4 = -3\sqrt{2} + (3\sqrt{2})i$

$$|z_4| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2}$$

$$= \sqrt{36}$$

$$= 6$$



$$\theta = \tan^{-1}\left(\frac{3\sqrt{2}}{3\sqrt{2}}\right)$$

$$\theta = 45^\circ$$

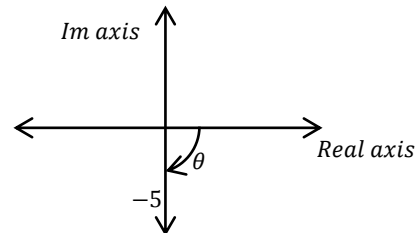
$$\arg(z_4) = +135^\circ$$

$$z_4 = 6(\cos 135 + i \sin 135)$$

(e) $z_5 = -5i = 0 + -5i$

$$|z_5| = \sqrt{0^2 + (-5)^2}$$

$$|z_5| = 5$$



$$\arg z_5 = -90$$

$$z_5 = 5(\cos -90 + i \sin -90)$$

$$z_6 = 3 + 4i$$

$$r = |z_6| = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{25}$$

$$= 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1$$

$$z_6 = 5(\cos 53.1^\circ + i \sin 53.1^\circ)$$

(g) $z_6 = -5 - 12i$

$$|z_6| = \sqrt{(-5)^2 + (-12)^2}$$

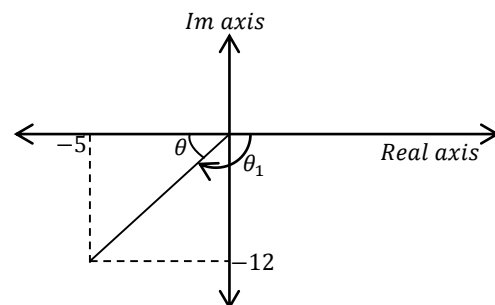
$$= \sqrt{169}$$

$$= 13$$

$$\theta = \tan^{-1}\left(\frac{12}{5}\right)$$

$$\theta = 67.4$$

$$z_7 = -5 - 12i$$



$$\arg z_7 = -112.6^\circ$$

$$|z_7| = \sqrt{(-5)^2 + (-12)^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

$$= 13$$

$$13(\cos -112.6 + i \sin -122.6)$$

Example II

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

Show that

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

$$\text{And } \frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

Solution

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) r_2(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 + \cos \theta_1 (i \sin \theta_2) \\ &\quad + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2)] \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\ &\quad + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2))] \\ &= r_1 r_2 [(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))] \end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} \\ &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_1(\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)} \\ &= \frac{r_1}{r_2} \left[\frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 + \sin^2 \theta_2} \right] \\ &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2} \right) \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))}{1} \quad (\text{as required})$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)$$

Example III

Given that $z_1 = 1 + i$

$$z_2 = \sqrt{3} - i$$

Find in polar form $z_1 z_2$ and $\frac{z_1}{z_2}$

Solution

$$\begin{aligned} z_1 &= 1 + i \\ |z_1| &= \sqrt{1^2 + 1^2} = \sqrt{2} \\ \arg z_1 &= \tan^{-1} \left(\frac{1}{1} \right) = \frac{\pi}{4} \\ z_1 &= r_1(\cos \theta_1 + i \sin \theta_1) \\ z_1 &= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ z_2 &= \sqrt{3} - i \\ |z_2| &= r_2 = \sqrt{(\sqrt{3})^2 + (-1)^2} \end{aligned}$$

$$= \sqrt{4}$$

$$= 2$$

$$\arg z_2 = -30^\circ$$

$$= -\frac{\pi}{6} \text{ radians}$$

$$\arg z_2 = -\frac{\pi}{6}$$

$$z_2 = 2 \left(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6} \right)$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \\ &= 2\sqrt{2} \left(\cos \left(\frac{\pi}{4} + -\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{4} + -\frac{\pi}{6} \right) \right) \end{aligned}$$

$$= 2\sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

$$= \frac{\sqrt{2}}{2} \left[\cos \left(\frac{\pi}{4} - \frac{-\pi}{6} \right) + i \sin \left(\frac{\pi}{4} - \frac{-\pi}{6} \right) \right]$$

$$= \frac{\sqrt{2}}{2} \left[\cos \left(\frac{5\pi}{12} \right) + i \sin \left(\frac{5\pi}{12} \right) \right]$$

Demoivre's Theorem

Demoivre's theorem states that for real values of n

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$$

Proving Demoivre's theorem by mathematical induction

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$$

For $n=1$, $(\cos \theta + i \sin \theta)^1 = (\cos \theta + i \sin \theta)$

It's true for $n=1$

Assume the results holds for the general value of $n=k$

$$(\cos \theta + i \sin \theta)^k = (\cos k\theta + i \sin k\theta)$$

It must be true for the next integer $n = k + 1$

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) \\ &= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta) \\ &= \cos k\theta \cos \theta + i \cos k\theta \sin \theta + i \sin k\theta \cos \theta \\ &\quad + i^2 \sin \theta \sin k\theta \\ &= [(\cos k\theta \cos \theta - \sin k\theta \sin \theta) + \\ &\quad i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)] \\ &= \cos(k\theta + \theta) + i \sin(k\theta + \theta) \\ &= \cos(k + 1)\theta + i \sin(k + 1)\theta \end{aligned}$$

$$\begin{aligned} &\Rightarrow (\cos \theta + i \sin \theta)^{k+1} \\ &= \cos(k+1)\theta + i \sin(k+1)\theta \end{aligned}$$

For the next integer, $n = k + 1 = 2$

$$\Rightarrow k = 1$$

$$(\cos \theta + i \sin \theta)^2 = (\cos 2\theta + i \sin 2\theta)$$

Since it's true for $n=1$, $n=2$ and so on it's true for all positive integral values of n .

Example I

Find the value of $(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)^{12}$

Solution

$$\begin{aligned} (\cos \theta + i \sin \theta)^n &= \cos n\theta + i \sin n\theta \\ \left(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi\right)^{12} &= \\ \left(\cos \frac{1}{4}\pi \times 12 + i \sin \frac{1}{4}\pi \times 12\right) &= \\ = \cos 3\pi + i \sin 3\pi &= \\ = -1 & \end{aligned}$$

Example II

Express $(1 - i\sqrt{3})^4$ in the form $a + bi$

Solution

$$(1 - i\sqrt{3})^4$$

$$\text{Let } z = 1 - i\sqrt{3}$$

$$|z| = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$$

$$\begin{aligned} \arg z &= -60 \\ &= -\frac{\pi}{3} \end{aligned}$$

$$\arg z = -\frac{\pi}{3}$$

$$z = r(\cos \theta + i \sin \theta)$$

$$z = 2\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)$$

$$z^4 = 2^4\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)^4$$

$$= 16\left(\cos -\frac{4\pi}{3} + i \sin -\frac{4\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -8 + 8\sqrt{3}i$$

Example III

Evaluate $\frac{1}{(1-i\sqrt{3})^3}$

Solution:

$$\frac{1}{1-i\sqrt{3}} = (1-i\sqrt{3})^{-3}$$

$$\text{Let } z = (1-i\sqrt{3})$$

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2}$$

$$= 2$$

$$\arg z = -\frac{\pi}{3}$$

$$(1-i\sqrt{3}) = 2\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)$$

$$(1-i\sqrt{3})^{-3} = 2^{-3}\left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)^{-3}$$

$$= \frac{1}{8}\left(\cos -\frac{\pi}{3} \times -3 + i \sin -\frac{\pi}{3} \times -3\right)$$

$$= \frac{1}{8}(\cos + \pi + i \sin + \pi)$$

$$= -\frac{1}{8}$$

Example IV

Express $\sqrt{3} + i$ in modulus -argument form. Hence find

$$(\sqrt{3} + i)^{10} \text{ and } \frac{1}{(\sqrt{3} + i)^7} \text{ in the form of } a + bi$$

Solution

$$\text{Let } z = \sqrt{3} + i$$

$$|z| = \sqrt{(\sqrt{3})^2 + 1} = 2$$

$$\arg z = \frac{\pi}{6}$$

$$z = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

$$(\sqrt{3} + i)^{10} = 2^{10}\left(\cos \left(\frac{10\pi}{6}\right) + i \sin \left(\frac{10\pi}{6}\right)\right)$$

$$= 2^{10}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{1024}{2} - \frac{i(1024)\sqrt{3}}{2}$$

$$= 512 - 512\sqrt{3}i$$

$$\frac{1}{(\sqrt{3} + i)^7} = (\sqrt{3} + i)^{-7}$$

$$= \left(2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right)^{-7}$$

$$= 2^{-7}\left(\cos -7 \times \frac{\pi}{6} + i \sin -7 \times \frac{\pi}{6}\right)$$

$$\begin{aligned}
&= \frac{1}{128} (\cos \frac{-7\pi}{6} + i \sin \frac{-7\pi}{6}) \\
&= \frac{1}{128} \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\
&= -\frac{\sqrt{3}}{256} + \frac{1}{256}i \\
&= -\frac{\sqrt{3}}{256} + \frac{1}{256}i
\end{aligned}$$

Example V

Express $(-1+i)$ in modulus – argument form. Hence show that $(-1+i)^{16}$ is real and that

$\frac{1}{(-1+i)^6}$ is pure imaginary.

Solution

$$z = -1 + i$$

$$|z| = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

$$\arg z = 135^\circ$$

$$z = \sqrt{2}(\cos 135 + i \sin 135)$$

$$z^{16} = (\sqrt{2})^{16} (\cos 135 \times 16 + i \sin 135 \times 16)$$

$$= 256(\cos 2160 + i \sin 2160)$$

$$= 256(1)$$

$$= 256$$

$$\Rightarrow (-1+i)^{16} = 256 \text{ So it is purely real}$$

As required

$$\frac{1}{(-1+i)^6} = (-1+i)^{-6}$$

$$z^{-6} = (\sqrt{2})^{-6} (\cos 135 \times -6 + i \sin 135 \times -6)$$

$$= \frac{1}{8} (0 + i) = \frac{1}{8}i$$

$$\Rightarrow z^{-6} \text{ is purely imaginary.}$$

Example VI

a) $(\cos \theta + i \sin \theta)^2 (\cos \theta + i \sin \theta)^3$

b) $\frac{1}{(\cos \theta + i \sin \theta)^2}$

c) $\frac{\cos \theta + i \sin \theta}{(\cos \theta + i \sin \theta)^4}$

d) $\frac{(\cos \frac{\pi}{17} + i \sin \frac{\pi}{17})^8}{(\cos \frac{\pi}{17} - i \sin \frac{\pi}{17})^9}$

e) $\frac{(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta)}{(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})}$

f) $\frac{(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^8}{(\cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5})^3}$

Solutions

(a) $(\cos \theta + i \sin \theta)^2 (\cos \theta + i \sin \theta)^3$
 $= (\cos \theta + i \sin \theta)^{2+3}$
 $= (\cos \theta + i \sin \theta)^5$
 $= (\cos 5\theta + i \sin 5\theta)$

(b) $\frac{1}{(\cos \theta + i \sin \theta)^2}$
 $= (\cos \theta + i \sin \theta)^{-2}$
 $= \cos -2\theta + i \sin -2\theta$
 $= \cos 2\theta - i \sin 2\theta$

(c) $\frac{\cos \theta + i \sin \theta}{(\cos \theta + i \sin \theta)^4}$
 $= (\cos \theta + i \sin \theta)^1 (\cos \theta + i \sin \theta)^{-4}$
 $= (\cos \theta + i \sin \theta)^{1-4}$
 $= (\cos \theta + i \sin \theta)^{-3}$
 $= \cos -3\theta + i \sin -3\theta$
 $= (\cos 3\theta - i \sin 3\theta)$

(d) $\frac{(\cos \frac{\pi}{17} + i \sin \frac{\pi}{17})^8}{(\cos \frac{\pi}{17} - i \sin \frac{\pi}{17})^9}$
 $\frac{(\cos \pi + i \sin \pi)^{\frac{8}{17}}}{(\cos \pi + i \sin \pi)^{-\frac{9}{17}}}$
 $(\cos \pi + i \sin \pi)^{\frac{8}{17} - \frac{-9}{17}}$
 $(\cos \pi + i \sin \pi)^1$

(e) $\frac{(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta)}{(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})}$
 $\frac{(\cos \theta + i \sin \theta)(\cos \theta + i \sin \theta)^2}{(\cos \theta + i \sin \theta)^{\frac{1}{2}}}$
 $\frac{(\cos \theta + i \sin \theta)^3}{(\cos \theta + i \sin \theta)^{\frac{1}{2}}}$
 $(\cos \theta + i \sin \theta)^{\frac{5}{2}}$
 $\cos \frac{5}{2}\theta + i \sin \frac{5}{2}\theta$

(f) $\frac{(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^8}{(\cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5})^3}$
 $\frac{\left((\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})^2 \right)^8}{\left((\cos \frac{\pi}{5} + i \sin \frac{\pi}{5})^{-3} \right)^3}$

$$\frac{\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)^{16}}{\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)^{-9}}$$

$$\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)^{16-(-9)}$$

$$\cos\left(25 \times \frac{\pi}{5}\right) + i \sin\left(25 \times \frac{\pi}{5}\right)$$

$$= (\cos 5\pi + i \sin 5\pi)$$

Example VII

Use De-moivre's theorem to show that

$$\tan 3\theta = \frac{3 \tan \theta - 3 \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

Solution

$$(\cos 3\theta + i \sin 3\theta) = (\cos \theta + i \sin \theta)^3$$

$$\text{but } (\cos \theta + i \sin \theta)^3 =$$

$$= \cos^3 \theta + 3(i \sin \theta) \cos^2 \theta + 3(i \sin \theta)^2 \cos \theta$$

$$+ (i \sin \theta)^3$$

$$= (\cos^3 \theta - 3 \sin^2 \theta \cos \theta) +$$

$$i(3 \sin \theta \cos^2 \theta - \sin^3 \theta)$$

$$= \cos 3\theta + i \sin 3\theta$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow \sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta \dots (1)$$

$$\cos 3\theta = \cos^3 \theta - 3 \sin^2 \theta \cos \theta \dots \dots \dots (2)$$

Eqn (1) ÷ Eqn (2)

$$\Rightarrow \tan 3\theta = \frac{3 \sin \theta \cos^2 \theta - \sin^3 \theta}{\cos^3 \theta - 3 \sin^2 \theta \cos \theta}$$

$$\tan 3\theta = \frac{\frac{3 \sin \theta \cos^2 \theta}{\cos^3 \theta} - \frac{\sin^3 \theta}{\cos^3 \theta}}{\frac{\cos^3 \theta}{\cos^3 \theta} - \frac{3 \sin^2 \theta \cos \theta}{\cos^3 \theta}}$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

Example VIII

Use Demovre's theorem to show that

$$\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

Solution

$$(\cos \theta + i \sin \theta)^4 = (\cos \theta + i \sin \theta)^4$$

$$= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 +$$

$$4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4$$

$$= \cos^4 \theta + (4 \cos^3 \theta \sin \theta)i - 6 \cos^2 \theta \sin^2 \theta$$

$$- (4 \cos \theta \sin^3 \theta)i + \sin^4 \theta$$

$$= (\cos 4\theta + i \sin 4\theta)$$

Equating real to real and imaginary to imaginary

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \dots (i)$$

$$\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta \dots \dots (ii)$$

Eqn (ii) ÷ Eqn (i)

$$\tan 4\theta = \frac{4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta}$$

$$\tan 4\theta = \frac{\frac{4 \cos^3 \theta \sin \theta}{\cos^4 \theta} - \frac{4 \cos \theta \sin^3 \theta}{\cos^4 \theta}}{\frac{\cos^4 \theta}{\cos^4 \theta} - \frac{6 \cos^2 \theta \sin^2 \theta}{\cos^4 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta}}$$

$$\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

Example IX

Show that

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

Hence show that $\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$

Solution

$$z = \cos \theta + i \sin \theta$$

$$z^n = (\cos \theta + i \sin \theta)^n$$

$$= (\cos n\theta + i \sin n\theta)$$

$$z^{-n} = (\cos \theta + i \sin \theta)^{-n}$$

$$= \cos -n\theta + i \sin -n\theta$$

$$= \cos n\theta - i \sin n\theta$$

$$z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$$

$$= 2 \cos n\theta$$

$$z^n - \frac{1}{z^n}$$

$$= (\cos n\theta + i \sin n\theta) - (\cos n\theta - i \sin n\theta)$$

$$= 2i \sin n\theta$$

$$\text{from } z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

$$z - \frac{1}{z} = 2i \sin \theta$$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$\left(z + \frac{1}{z}\right)^4 = (2 \cos \theta)^4$$

But $\left(z + \frac{1}{z}\right)^4$

$$z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = \left(z + \frac{1}{z}\right)^4$$

$$2 \cos 4\theta + 4(2 \cos 2\theta) + 6 = (2 \cos \theta)^4$$

$$16 \cos^4 \theta = 2 \cos 4\theta + 4(2 \cos 2\theta) + 6$$

$$\cos^4 \theta = \frac{1}{16}(2 \cos 4\theta + 8 \cos 2\theta + 6)$$

$$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3)$$

Example XI

Given that $z = \cos \theta + i \sin \theta$ show that

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

Hence or otherwise show that

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

Solution

$$z^n - \frac{1}{z^n} = 2i \sin \theta$$

$$z - \frac{1}{z} = 2i \sin \theta$$

$$\left(z - \frac{1}{z}\right)^5 = (2i \sin \theta)^5$$

$$\left(z - \frac{1}{z}\right)^5 = i^5(32) \sin^5 \theta$$

$$\left(z - \frac{1}{z}\right)^5 = (32i \times i^4 \sin^5 \theta)$$

$$\left(z - \frac{1}{z}\right)^5 = 32i \sin^5 \theta$$

$$\text{but } \left(z - \frac{1}{z}\right)^5 = z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2$$

$$+ 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$z^n - \frac{1}{z^n} = 2i \sin n\theta$$

$$z^5 - \frac{1}{z^5} = 2i \sin 5\theta$$

$$z - \frac{1}{z} = 2i \sin \theta$$

$$\left(z - \frac{1}{z}\right)^5 = (2i \sin \theta)^5$$

$$2i \sin 5\theta - 5(2i \sin 3\theta) + 10(2i \sin \theta) = 32i \sin^5 \theta$$

$$\sin^5 \theta = \frac{1}{32}(2 \sin 5\theta - 10 \sin 3\theta + 20 \sin \theta)$$

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

Example XII

Prove that $\cos^6 \theta + \sin^6 \theta = \frac{1}{8}(3 \cos 4\theta + 5)$

Solution

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$\left(z + \frac{1}{z}\right)^6 = (2 \cos \theta)^6$$

$$\left(z + \frac{1}{z}\right)^6 = 64 \cos^6 \theta$$

$$\text{But } \left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z}\right)^2 +$$

$$20z^3\left(\frac{1}{z}\right)^3 + 15z^2\left(\frac{1}{z}\right)^4 + 6z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= \left(z^6 + \frac{1}{z^6}\right) + \left(6z^4 + \frac{6}{z^4}\right) + \left(15z^2 + \frac{15}{z^2}\right) + 20$$

$$= 2 \cos 6\theta + 6(2 \cos 4\theta) + 15(2 \cos 2\theta) + 20$$

$$\Rightarrow 64 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$$

$$64 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta$$

$$+ 20 \dots \dots \dots (1)$$

$$\left(z - \frac{1}{z}\right) = 2i \sin \theta$$

$$\left(z - \frac{1}{z}\right)^6 = 64i^6 \sin^6 \theta$$

$$\left(z - \frac{1}{z}\right)^6 = -64 \sin^6 \theta$$

But

$$\left(z - \frac{1}{z}\right)^6 = z^6 + 6z^5\left(-\frac{1}{z}\right) + 15z^4\left(-\frac{1}{z}\right)^2 +$$

$$20z^3\left(-\frac{1}{z}\right)^3 + 15z^2\left(-\frac{1}{z}\right)^4 + 6z\left(-\frac{1}{z}\right)^5 + \left(-\frac{1}{z}\right)^6$$

$$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2 \cos 6\theta - 6(2 \cos 4\theta) + 15(2 \cos 2\theta) - 20$$

$$\Rightarrow 2 \cos 6\theta - 12 \cos 4\theta + 30 \cos 2\theta - 20$$

$$= -64 \sin^6 \theta$$

$$-64 \sin^6 \theta = 2 \cos 6\theta - 12 \cos 4\theta + 30 \cos 2\theta - 20 \dots \dots \dots (2)$$

Eqn (2) – Eqn (1)

$$\Rightarrow 64 \cos^6 \theta - -64 \sin^6 \theta = 24 \cos 4\theta + 40$$

$$\cos^6 \theta + \sin^6 \theta = \frac{8}{64} (3 \cos 4\theta + 5)$$

$$\cos^6 \theta + \sin^6 \theta = \frac{1}{8} (3 \cos 4\theta + 5)$$

Solving Complex Equations

Given that x and y are real numbers. Find the values of x and y which satisfy the equation.

$$\frac{2y + 4i}{2x + y} - \frac{y}{x - i} = 0$$

Solution

$$\frac{2y + 4i}{2x + y} - \frac{y}{x - i} = 0$$

$$\frac{2y + 4i}{2x + y} = \frac{y}{x - i}$$

$$\frac{2y + 4i}{2x + y} = \frac{y}{x - i} \times \frac{x + i}{x + i}$$

$$\frac{2y + 4i}{2x + y} = \frac{xy + iy}{x^2 + 1}$$

$$\frac{2y}{2x + y} + \frac{4i}{2x + y} = \frac{xy}{x^2 + 1} + \frac{yi}{x^2 + 1}$$

Equating real to real and imaginary to imaginary

$$\Rightarrow \frac{2y}{2x + y} = \frac{xy}{x^2 + 1} \dots \dots \dots (1)$$

$$\frac{4}{2x + y} = \frac{y}{x^2 + 1} \dots \dots \dots (2)$$

From equation (1)

$$2y(x^2 + 1) = xy(2x + y)$$

$$2x^2y + 2y = 2x^2y + xy^2$$

$$2y - xy^2 = 0$$

$$y(2 - xy) = 0$$

$$y = 0 \text{ or } xy = 2$$

From Eqn (2), $\frac{4}{2x + y} = \frac{y}{x^2 + 1}$

$$4x^2 + 4 = 2xy + y^2$$

For $y = 0$, $4x^2 + 4 = 0$

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x^2 = i^2$$

$$x = \pm i$$

For $xy = 2$, $y = \frac{2}{x}$

$$\Rightarrow 4x^2 + 4 = 4 + \frac{4}{x^2}$$

$$4x^2 - \frac{4}{x^2} = 0$$

Let $x^2 = m$

$$4m - \frac{4}{m} = 0$$

$$4m^2 - 4 = 0$$

$$m^2 - 1 = 0$$

$$(m + 1)(m - 1) = 0$$

$$m = 1, m = -1$$

When $m = 1$, $x^2 = 1 \Rightarrow x = \pm 1$

When $m = -1$, $x^2 = i^2 \Rightarrow x = \pm i$

$$xy = 2$$

$$\text{If } x = 1, y = 2$$

$$\text{If } x = -1, y = -2$$

$$\text{If } x = i, y = -2i$$

$$\text{If } x = -i, y = 2i$$

Example II

Find the values of x and y in

$$\frac{x}{2 + 3i} - \frac{y}{3 - 2i} = \frac{6 + 2i}{1 + 8i}$$

Solution

$$\frac{x}{2 + 3i} - \frac{y}{3 - 2i} = \frac{6 + 2i}{1 + 8i}$$

$$\frac{x(2 - 3i)}{2 + 3i(2 - 3i)} - \frac{y(3 + 2i)}{(3 - 2i)(3 + 2i)} = \frac{(6 + 2i)(1 - 8i)}{(1 + 8i)(1 - 8i)}$$

$$\frac{(2x - 3xi)}{13} - \frac{(3y + 2yi)}{13} = \frac{(6 - 48i + 2i + 16)}{65}$$

$$\frac{2x - 3y}{13} - \frac{3x + 2y}{13}i = \frac{22}{65} - \frac{46}{65}i$$

$$\Rightarrow \frac{2x - 3y}{13} = \frac{22}{65}$$

$$5(2x - 3y) = 22$$

$$10x - 15y = 22 \dots \dots \dots (1)$$

Similarly, $\frac{3x + 2y}{13} = \frac{46}{65}$

$$5(3x + 2y) = 46$$

$$15x + 10y = 46 \dots \dots \dots (2)$$

Solving eqn (1) and eqn (2) simultaneously

$$\Rightarrow x = 2.8 \quad y = 0.4$$

Example III

Find the values of x and y if $\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$

Solution

$$\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$$

$$\frac{x(1-i)}{(1+i)(1-i)} + \frac{y(2+i)}{(2-i)(2+i)} = 2 + 4i$$

$$\frac{x-xi}{2} + \frac{2y+yi}{5} = 2 + 4i$$

$$5(x-xi) + 2(2y+yi) = 2 + 4i$$

$$5x - 5xi + 4y + 2yi = 20 + 40i$$

Equating real to real and imaginary to imaginary;

$$5x + 4y = 20 \dots\dots\dots (1)$$

$$2y - 5x = 40 \dots\dots\dots (2)$$

Solving Eqn (1) and Eqn (2) simultaneously;

$$y = 10$$

$$x = -4$$

Example IV

Find the values of x and y. given that

$$\frac{xi}{1+iy} = \frac{3x+4i}{x+3y}$$

Solution

$$\frac{xi(1-iy)}{(1+iy)(1-iy)} = \frac{(3x+4i)}{x+3y}$$

$$\frac{xi+xy}{1+y^2} = \frac{3x+4i}{x+3y}$$

$$\frac{xy}{1+y^2} + \frac{xi}{1+y^2} = \frac{3x}{x+3y} + \frac{4i}{x+3y}$$

$$\Rightarrow \frac{xy}{1+y^2} = \frac{3x}{x+3y} \dots\dots\dots (1)$$

$$\frac{x}{1+y^2} = \frac{4}{x+3y} \dots\dots\dots (2)$$

From equation (1)

$$x^2y + 3xy^2 = 3x + 3xy^2$$

$$\Rightarrow x^2y = 3x$$

$$\Rightarrow x^2y - 3x = 0$$

$$x(xy - 3) = 0$$

$$x = 0 \text{ or } xy = 3$$

From eqn (2)

$$x^2 + 3xy = 4 + 4y^2 \dots\dots\dots (3)$$

When $x = 0$, $0 = 4 + 4y^2$

$$-1 = y^2$$

$$y = \pm i$$

When $xy = 3$

$$y = \frac{3}{x}$$

Substituting $y = \frac{3}{x}$ and $xy = 3$ in Eqn (3);

$$x^2 + 3(3) = 4 + 4\left(\frac{3}{x}\right)^2$$

$$x^2 + 9 = 4 + \frac{36}{x^2}$$

$$x^2 + 9 = 4 + \frac{36}{x^2}$$

$$x^2 - \frac{36}{x^2} + 5 = 0$$

$$\text{let } x^2 = P$$

$$P - \frac{36}{P} + 5 = 0$$

$$P^2 - 36 + 5P = 0$$

$$P^2 + 5P - 36 = 0$$

$$(P + 9)(P - 4) = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$$x^2 - 4 = 0 \Rightarrow x = \pm 2$$

$$x = 2, \quad x = -2$$

When $x = 2$, $y = \frac{3}{2}$

When $x = -2$, $y = -\frac{3}{2}$

$$x^2 + 9 = 0$$

$$x^2 = -9$$

$$x = \pm 3i$$

when $x = 3i$

$$y = \frac{3}{3i} = \frac{1}{i}$$

$$y = -i$$

when $x = -3i$

$$y = +i$$

Example V

If z is a complex number such that $z = \frac{P}{2-i} + \frac{q}{1+3i}$.

Where p and q are real. If $|z| = 7$, $\arg P = \frac{\pi}{2}$. Find the value of p and q .

Solution

$$z = \frac{P}{2-i} + \frac{q}{1+3i}$$

$$z = \frac{P(2+i)}{(2-i)(2+i)} + \frac{q(1-3i)}{(1+3i)(1-3i)}$$

$$z = \frac{2p + Pi}{5} + \frac{q - 3qi}{10}$$

$$z = \frac{2(2p + pi) + q - 3qi}{10}$$

$$z = \frac{4p + 2pi + q - 3qi}{10}$$

$$z = \frac{4p + q + (2p - 3q)i}{10}$$

$$\arg z = \tan^{-1} \left(\frac{\frac{2p - 3q}{10}}{\frac{4p + q}{10}} \right) = \frac{\pi}{2}$$

$$\tan^{-1} \left(\frac{2p - 3q}{4p + q} \right) = \frac{\pi}{2}$$

$$\frac{2p - 3q}{4p + q} = \infty$$

$$4p + q = 0$$

$$q = -4p$$

$$|z| = 7$$

$$\sqrt{\left(\frac{4p+q}{10}\right)^2 + \left(\frac{2p-3q}{10}\right)^2} = 7 \dots\dots\dots (1)$$

Substituting $q = -4p$ in Eqn (1)

$$\sqrt{0^2 + \left(\frac{14p}{10}\right)^2} = 7$$

$$\frac{14p}{10} = 7$$

$$p = 5$$

$$q = -4 \times 5$$

$$q = -20$$

Example VI

Given that $(1 + 5i)p - 2q = 3 + 7i$, find p and q

- (a) When p and q are real
- (b) When p and q are conjugate complex numbers

Solution

(a) $(1 + 5i)p - 2q = 3 + 7i$

$$p + 5Pi - 2q = 3 + 7i$$

$$p - 2q + 5Pi = 3 + 7i$$

$$p - 2q = 3 \dots\dots\dots (1)$$

$$5p = 7 \dots\dots\dots (2)$$

From Eqn (2),

$$p = \frac{7}{5}$$

$$\Rightarrow \frac{7}{5} - 2q = 3$$

$$\frac{7}{5} - 3 = 2q$$

$$-\frac{8}{5} = 2q$$

$$-\frac{8}{10} = q$$

$$q = -\frac{4}{5}$$

$$\Rightarrow p = \frac{7}{5}, \quad q = -\frac{4}{5}$$

(b) Let $p = x + iy$

$$q = x - iy$$

$$(1 + 5i)(x + iy) - 2(x - iy) = 3 + 7i$$

$$x + iy + 5xi - 5y - 2x + 2yi = 3 + 7i$$

$$(x - 5y - 2x) + (y + 5x + 2y)i = 3 + 7i$$

$$(-x - 5y) + (3y + 5x)i = 3 + 7i$$

$$-x - 5y = 3$$

$$x = -3 - 5y \dots\dots\dots (1)$$

$$3y + 5x = 7 \dots\dots\dots (2)$$

Substituting Eqn (1) in Eqn (2)

$$3y + 5(-3 - 5y) = 7$$

$$3y - 15 - 25y = 7$$

$$-22y = 22$$

$$y = -1$$

$$x = -3 - 5(-1)$$

$$x = -3 + 5$$

$$x = 2$$

$$p = x + iy$$

$$p = 2 - i$$

$$q = 2 + i$$

Square root of Complex Numbers

Example I

Find the square root of $35 - 12i$

Solution

Let $\sqrt{35 - 12i} = x + iy$

$$(\sqrt{35 - 12i})^2 = (x + iy)^2$$

$$35 - 12i = x^2 + 2xyi + i^2y^2$$

$$35 - 12i = x^2 - y^2 + 2xyi$$

$$\Rightarrow x^2 - y^2 = 35$$

$$2xy = -12$$

$$xy = -6$$

$$y = -\frac{6}{x}$$

$$x^2 - \frac{36}{x^2} = 35$$

$$x^4 - 36 = 35x^2$$

$$x^4 - 35x^2 - 36 = 0$$

Let $x^2 = m$

$$m^2 - 35m - 36 = 0$$

$$(m - 36)(m + 1) = 0$$

$$(x^2 - 36)(x^2 + 1) = 0$$

But x is real

$$\Rightarrow x^2 - 36 = 0$$

$$x = \pm 6$$

When $x = 6$ $y = -\frac{6}{6}$

$$y = -1$$

when $x = -6$, $y = 1$

$$\Rightarrow \sqrt{35 - 12i} = 6 - i$$

or $\sqrt{35 - 12i} = -6 + i$

Example VIII

Find the square root of $5 - 12i$

solution

Let $\sqrt{5 - 12i} = x + iy$

$$5 - 12i = (x + iy)^2$$

$$5 - 12i = x^2 + 2xyi + yi^2$$

$$5 - 12i = x^2 - y^2 + 2xyi$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow x^2 - y^2 = 5 \dots\dots\dots(1)$$

$$2xy = -12$$

$$xy = -6$$

$$y = -\frac{6}{x} \dots\dots\dots(2)$$

Substituting Eqn (2) in Eqn (1)

$$x^2 - \frac{36}{x^2} = 5$$

$$(x^2)^2 - 36 = 5x^2$$

let $m = x^2$

$$m^2 - 36 = 5m$$

$$m^2 - 5m - 36 = 0$$

$$(m - 9)(m + 4) = 0$$

$$(x^2 - 9)(x^2 + 4) = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

when $x = 3$, $y = -2$

when $x = -3$, $y = 2$

$$\sqrt{5 - 12i} = 3 - 2i$$

or $\sqrt{5 - 12i} = 3 + 2i$

Example IX

Find the roots of $z^2 - (1 - i)z + 7i - 4 = 0$

Solution

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{(1 - i) \pm \sqrt{(1 - i)^2 - 4(1)(7i - 4)}}{2 \times 1}$$

$$z = \frac{1 - i \pm \sqrt{1 - 2i - 1 - 28i + 16}}{2}$$

$$\frac{1 - i \pm \sqrt{16 - 30i}}{2}$$

But $\sqrt{16 - 30i} = a + bi$

$$16 - 30i = a^2 + 2abi - b^2$$

$$a^2 - b^2 = 16$$

$$2ab = -30$$

$$ab = -15$$

$$a = \frac{-15}{b}$$

$$\left(\frac{15}{b}\right)^2 - b^2 = 16$$

$$\frac{225}{b^2} - b^2 = 16$$

Let $m = b^2$

$$\frac{225}{m} - m = 16$$

$$m^2 + 16m - 225 = 0$$

$$m = 9, \quad m = -25$$

$$b^2 = 9$$

$$b = \pm 3$$

$$ab = -15$$

$$a = 5$$

When $b = -3$, $a = 5$

When $b = 3$, $a = -5$

$$a + bi = 5 - 3i, \quad -5 + 3i$$

$$\sqrt{16 - 30i} = \pm(5 - 3i)$$

$$z = \frac{1 - i \pm (5 - 3i)}{2}$$

$$z = 3 - 2i$$

$$z = -2 + i$$

Example X

Show that $1 + 2i$ is a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

Solution

$$z = 1 + 2i$$

$$z^2 = (1 + 2i)^2$$

$$= 1 + 4i + 4i^2$$

$$= -3 + 4i$$

$$z^3 = z \times z^2 = (1 + 2i)(-3 + 4i)$$

$$= -3 + 4i - 6i - 8$$

$$= -11 - 2i$$

$$\begin{aligned}
& 2z^3 - z^2 + 4z + 15 \\
\Rightarrow & 2(-11 - 2i) - (-3 + 4i) + (4(1 + 2i) + 15) \\
& = -22 - 4i + 3 - 4i + 4 + 8i + 15 \\
& = -22 + 22 - 8i + 8i \\
& = 0 + 0i \\
& = 0
\end{aligned}$$

$\Rightarrow 1 + 2i$ is a root of the equation.

Since $z = 1 + 2i$ is a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

The complex conjugate $\bar{z} = 1 - 2i$ must also be a root of the above equation

$\Rightarrow 1 - 2i = z$ is also a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

$$2z^3 - z^2 + 4z + 15 = 0$$

$$z = 1 + 2i$$

$$z = 1 - 2i$$

Sum of roots = $1 + 2i + 1 - 2i$

$$= 2$$

Product of roots = $(1)^2 - (2i)^2$

$$= 1 + 4$$

$$= 5$$

$$z^2 - \left(\begin{matrix} \text{sum of} \\ \text{roots} \end{matrix}\right)z + \text{product} = 0$$

$$z^2 - 2z + 5 = 0$$

$\Rightarrow z^2 - 2z + 5$ is a factor of $2z^3 - z^2 + 4z + 15$

$$\begin{array}{r}
z^2 - 2z + 5 \overline{) 2z^3 - z^2 + 4z + 15} \\
\underline{2z^3 - 4z^2 + 10z} \\
3z^2 - 6z + 15 \\
\underline{3z^2 - 6z + 15} \\
0
\end{array}$$

$$(2z + 3)(z^2 - 2z + 5) = 0$$

$$z = -\frac{3}{2} \quad z = 1 + 2i \text{ and } z = 1 - 2i$$

Example XI

Given that $2 + 3i$ is a root of the equation

$z^3 - 6z^2 + 21z - 26 = 0$. Find the other roots

Solution

$z = 2 + 3i$ is a root $\Rightarrow z = 2 - 3i$ is also a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$

Sum of roots = $2 + 3i + 2 - 3i$

$$= 4$$

Product of roots = $(2 + 3i)(2 - 3i)$

$$= 2^2 - (3i)^2$$

$$= 4 + 9$$

$$= 13$$

$\Rightarrow z^2 - 4z + 13$ is a factor of

$$z^3 - 6z^2 + 21z - 26 = 0$$

$$\begin{array}{r}
z^2 - 4z + 13 \overline{) z^3 - 6z^2 + 21z - 26} \\
\underline{z^3 - 4z^2 + 13z} \\
-2z^2 + 8z - 26 \\
\underline{-2z^2 + 8z - 26} \\
0
\end{array}$$

$$\Rightarrow (z - 2)(z^2 - 4z + 13) = 0$$

$\Rightarrow z = 2, z = 2 + 3i$ and $z = 2 - 3i$ are roots of equation of $z^3 - 6z^2 + 21z - 26 = 0$

Example XII

Show that $1 + i$ is a root of the equation

$z^4 + 3z^2 - 6z + 10 = 0$. Hence find other roots

Solution

$$z = 1 + i$$

$$z^2 = 1 + 2i + i^2$$

$$z^2 = 1 + 2i - 1$$

$$z^2 = 2i$$

$$z^3 = z^2 \cdot z$$

$$= 2i(1 + i)$$

$$= 2i - 2$$

$$z^4 = (z^2)^2 = (2i)^2 = 4i^2$$

$$= -4$$

$$\Rightarrow z^4 + 3z^2 - 6z + 10$$

$$= (-4) + 3(2i) - 6(1 + i) + 10$$

$$= -4 + 6i - 6 - 6i + 10$$

$$= -10 + 10 + 6i - 6i$$

$$= 0 + 0i$$

$$= 0$$

$z = 1 + i$ is a root of the equation

$\Rightarrow 1 - i$ is also a root of the equation

Sum of the roots = $1 + i + 1 - i$

$$= 2$$

Product of roots = $(1)^2 - i^2 = 2$

$$z^2 - (\text{sum of roots})z + \text{product} = 0$$

$$z^2 - 2z + 2 = 0$$

$\Rightarrow z^2 - 2z + 2$ is a factor of $z^4 + 3z^2 - 6z + 10$

$$\begin{array}{r} z^2 + 2z + 5 \\ \hline z^2 - 2z + 2 \overline{) z^4 + 3z^2 - 6z + 10} \\ \underline{z^4 - 2z^3 + 2z^2} \\ 2z^3 + z^2 - 6z + 10 \\ \underline{2z^3 - 4z^2 + 4z} \\ 5z^2 - 10z + 10 \\ \underline{5z^2 - 10z + 10} \\ 0 \end{array}$$

$$(z^2 - 2z + 2)(z^2 + 2z + 5) = 0$$

$$\Rightarrow z^2 + 2z + 5 = 0$$

$$z^2 - 2z + 2 = 0$$

For $z^2 + 2z + 5 = 0$, $z = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 1 \times 5}}{4 \times 1}$

$$z = \frac{-2 \pm \sqrt{16i^2}}{2}$$

$$z = \frac{-1 \pm 4i}{2}$$

$$z = -1 + 2i$$

$$z = -1 - 2i$$

$\Rightarrow -1 + 2i, -1 - 2i, 1 + i, 1 - i$ are roots of the equation $z^4 + 3z^2 - 6z + 10 = 0$

Example XIII

Show that $1 - i$ is a root of the equation $4z^4 - 8z^3 + 9z^2 - 2z + 2 = 0$. Find the other roots.

Solution

$$z = 1 - i$$

$$z^2 = (1 - i)^2$$

$$= 1 - 2i + i^2$$

$$= -2i$$

$$z^3 = z^2 \cdot z$$

$$= -2i(1 - i)$$

$$= -2i + 2i^2$$

$$= -2 - 2i$$

$$z^4 = (z^2)^2 = (-2i)^2$$

$$= -4$$

$$4z^4 - 8z^3 + 9z^2 - 2z + 2 =$$

$$4(-4) - 8(-2 - 2i) + 9(-2i) - 2(1 - i) + 2$$

$$= -16 + 16 + 16i - 18i - 2 + 2i + 2$$

$$= 0 + 0i = 0$$

Since $z = 1 - i$ is a root of the equation and it implies that $1 + i$ is also a root.

$$\text{Sum of roots} = 1 - i + 1 + i$$

$$= 2$$

$$\text{Product of the roots} = (1 + i)(1 - i)$$

$$1^2 - i^2 = 2$$

$$\Rightarrow z^2 - (2z) + 2 = 0$$

$\Rightarrow z^2 - 2z + 2$ is a factor of

$$z^4 - 8z^3 + 9z^2 - 2z + 2 = 0.$$

$$\begin{array}{r} 4z^2 + 1 \\ \hline z^2 - 2z + 2 \overline{) 4z^4 - 8z^3 + 9z^2 - 2z + 2} \\ \underline{4z^4 - 8z^3 + 8z^2} \\ z^2 - 2z + 2 \\ \underline{z^2 - 2z + 2} \\ 0 \end{array}$$

$$(z^2 - 2z + 1)(4z^2 + 1) = 0$$

$$4z^2 = -1 \quad z^2 = -\frac{1}{4}$$

$$\Rightarrow z^2 = \frac{1}{4}i^2, \quad 2 = \pm \frac{1}{2}i$$

Example XIV

Given that $z = 2 - i$ is a root of the equation $z^3 - 3z^2 + z + k = 0$, k is real. Find other roots.

Solution

$$z = 2 - i$$

$$z^2 = (2 - i)^2$$

$$= 4 - 4i + i^2$$

$$= 3 - 4i$$

$$z^3 = (2 - i)(3 - 4i)$$

$$= 6 - 8i - 3i + 4i^2$$

$$= 2 - 11i$$

$$\Rightarrow (2 - 11i) - 3(3 - 4i) + 2 - i + k = 0$$

$$2 - 11i - 9 + 12i + 2 - i + k = 0$$

$$-11i + 11i + 4 - 9 + k = 0$$

$$0 - 5 + k = 0$$

$$k = 5$$

$$\Rightarrow z^3 - 3z^2 + z + 5 = 0$$

$$z = 2 - i$$

$$z = 2 + i$$

$$z = 2 - i$$

$$z = 2 + i$$

Sum of roots = 4

Product of roots = 5

$$z^2 - 4z + 5 = 0 \text{ is a factor of}$$

$$z^3 - 3z^2 + z + 5 = 0$$

$$\begin{array}{r} z + 1 \\ \hline z^2 - 4z + 5 \overline{) z^3 - 3z^2 + z + 5} \\ \underline{z^3 - 4z^2 + 5z} \\ z^2 - 4z + 5 \end{array}$$

$$(z + 1)(z^2 - 4z + 5) = 0$$

$(z + 1) = 0 \Rightarrow z = -1$
 $\Rightarrow z = -1, z = 2 + i, z = 2 - i$ are roots of the equation $z^3 - 3z^2 + z + k = 0$ where $k = 5$

Example XIV

Solve for z_1 and z_2 in the simultaneous equations below

$$z_1 + (1 - i)z_2 = 0$$

$$3z_2 - 3z_1 = 2 - 5i$$

Solution

$$z_1 + (1 - i)z_2 = 0 \dots\dots\dots (1)$$

$$3z_2 - 3z_1 = 2 - 5i \dots\dots\dots (2)$$

From eqn (1)

$$z_1 = -(1 - i)z_2$$

substitute in eqn (2)

$$3z_2 - 3[-(1 - i)z_2] = 2 - 5i$$

$$3z_2 + 3(1 - i)z_2 = 2 - 5i$$

$$3z_2 - 3iz_2 + 3z_2 = 2 - 5i$$

$$6z_2 - 3iz_2 = 2 - 5i$$

$$z_2(6 - 3i) = 2 - 5i$$

$$z_2 = \frac{2 - 5i}{6 - 3i}$$

$$z_2 = \frac{(2 - 5i)(6 + 3i)}{(6 - 3i)(6 + 3i)}$$

$$z_2 = \frac{12 + 6i - 30i + 15}{36 + 9}$$

$$z_2 = \frac{27 - 24i}{45}$$

$$z_2 = \frac{9 - 8i}{15}$$

$$z_2 = \frac{9}{15} - \frac{8i}{15}$$

$$z_1 = -(1 - i)z_2$$

$$z_1 = -\left((1 - i)\left(\frac{9 - 8i}{15}\right)\right)$$

$$z_1 = -\left(\frac{9 - 8i - 9i - 8}{15}\right)$$

$$z_1 = -\left(\frac{1 - 17i}{15}\right)$$

$$z_1 = \frac{-1 + 17i}{15}$$

$$z_1 = -\frac{1}{15} + \frac{17i}{15}$$

Example XV

Solve the equation $z^3 - 1$

Solution

$$z^3 - 1 = (z)^3 - (1)^3$$

$$= (z - 1)(z^2 + z + 1)$$

Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$z^3 - 1 = (z - 1)(z^2 + z + 1) = 0$$

$$z = 1$$

$$z^2 + z + 1 = 0$$

$$z = \frac{(-1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2 \times 1}$$

$$z = \frac{-1 \pm \sqrt{3i^2}}{2}$$

$$z = -\frac{1}{2} + \frac{(\sqrt{3})i}{2}$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z = 1, z = -\frac{1}{2} + \frac{\sqrt{3}i}{2}, z = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$$

Alternatively we can use Demovre's theorem

$$z^3 - 1 = 0$$

$$z^3 = 1$$

$$z^3 = 1 + 0i$$

$$z = (1 + 0i)^{\frac{1}{3}}$$

$$\text{let } P = 1 + 0i$$

$$|P| = \sqrt{1} = 1$$

$$\arg P = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$P = r[\cos(0) + i \sin(0)]$$

$$P = 1(\cos 0 + i \sin 0)$$

$$z = P^{\frac{1}{3}}$$

$$z = 1^{\frac{1}{3}}(\cos(0 + 360n) + i \sin(0 + 360n))$$

For $n = 0, 1, 2 \dots$

(Depending on the number of roots you want)

$$\text{For } n = 0, z = 1^{\frac{1}{3}}(\cos(0 + 360) + i \sin(0 + 360))^{\frac{1}{3}}$$

$$z = 1^{\frac{1}{3}}(\cos 120 + i \sin 120)$$

$$z = 1\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

$$\text{For } n = 1, z = 1^{\frac{1}{3}}[\cos(0 + 360 \times 1) + i \sin(0 + 360 \times 1)]^{\frac{1}{3}}$$

$$z = 1(\cos 120 + i \sin 120)$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\text{For } n = 2, z = 1^{\frac{1}{3}}[\cos(0 + 360 \times 2) + i \sin(0 + 360 \times 2)]^{\frac{1}{3}}$$

$$z = 1(\cos 240 + i \sin 240)$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\Rightarrow z = 1, \quad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Example XVI

$$\text{Solve: } z^3 + 27 = 0$$

Solution

$$z^3 + 3^3 = (z + 3)(z^2 + 3z + 9)$$

$$\text{From } a^3 + b^3 = (a + b)(a^2 + ab + b^2)$$

$$\Rightarrow z^3 + 3^3 = (z + 3)(z^2 + 3z + 9)$$

$$z = -3$$

$$z^2 + 3z + 9 = 0$$

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2}$$

$$z = \frac{-3 \pm \sqrt{27i^2}}{2}$$

$$z = -3, \quad z = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$$

$$z = \frac{-3}{2} - \frac{3\sqrt{3}i}{2}$$

Alternatively, we can use Demovre's theorem

$$z^3 + 27 = 0$$

$$z^3 = -27$$

$$z = (-27 + 0i)^{\frac{1}{3}}$$

$$\text{let } P = -27 + 0i$$

$$|P| = \sqrt{(-27)^2 + 0i}$$

$$= 27$$

$$\arg P = 180$$

$$P = 27(\cos 180 + i \sin 180)$$

$$z = P^{\frac{1}{3}} = 27^{\frac{1}{3}}(\cos 180 + i \sin 180)^{\frac{1}{3}}$$

$$z = 27^{\frac{1}{3}}(\cos(180 + 360n) + i \sin(180 + 360n))^{\frac{1}{3}}$$

$$\text{When } n = 0, z = 27^{\frac{1}{3}}[(\cos 180 + i \sin 180)]^{\frac{1}{3}}$$

$$z = 3(\cos 60 + i \sin 60)$$

$$= 3\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \frac{3}{2} + \frac{3\sqrt{3}i}{2}$$

$$\text{For } n = 1, z = 27^{\frac{1}{3}}[(\cos(180 + 360 \times 1) + i \sin(180 + 360 \times 1))]^{\frac{1}{3}}$$

$$z = 3(\cos 180 + i \sin 180)$$

$$z = -3$$

$$\text{For } n = 2, z = 27^{\frac{1}{3}}[(\cos(180 + 360 \times 2) + i \sin(180 + 360 \times 2))]^{\frac{1}{3}}$$

$$z = 3(\cos 300 + i \sin 300)$$

$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$= \frac{3}{2} - \frac{3\sqrt{3}i}{2}$$

$$z = -3, \quad z = \frac{3}{2} + \frac{3\sqrt{3}i}{2} \text{ and } z = \frac{3}{2} - \frac{3\sqrt{3}i}{2}$$

Example XVII

Solve the equation

$$z^4 + 1 = 0$$

$$z^4 = -1 + 0i$$

$$z^4 = (-1 + 0i)^{\frac{1}{4}}$$

$$\text{let } P = -1 + 0i$$

$$|P| = 1$$

$$\arg P = 180$$

$$P = 1(\cos 180 + i \sin 180)$$

$$z = P^{\frac{1}{4}} = 1^{\frac{1}{4}}[(\cos(180 + 360n))$$

$$+ i \sin(180 + 360n)]^{\frac{1}{4}}$$

$$\text{For } n = 0, z = 1^{\frac{1}{4}}(\cos 45 + i \sin 45)$$

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\text{For } n = 1$$

$$z = 1^{\frac{1}{4}}(\cos 540 + i \sin 540)^{\frac{1}{4}}$$

$$z = 1(\cos 135 + i \sin 135)$$

$$z = \frac{-\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$$

$$\text{For } n = 2, z = 1^{\frac{1}{4}}(\cos 900 + i \sin 900)^{\frac{1}{4}}$$

$$z = 1(\cos 225 + i \sin 225)$$

$$z = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\text{For } n = 3$$

$$z = 1^{\frac{1}{4}}(\cos 1260 + i \sin 1260)^{\frac{1}{4}}$$

$$z = 1(\cos 315 + i \sin 315)$$

$$z = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$\text{For } z^4 + 1 = 0$$

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

Example XVIII

Find the fourth roots of -16

Solution

$$z = (-16)^{\frac{1}{4}} = (-16 + 0i)^{\frac{1}{4}}$$

$$\text{Let } P = -16 + 0i$$

$$z = P^{\frac{1}{4}} = (-16 + 0i)^{\frac{1}{4}}$$

$$|P| = 16$$

$$\arg P = 180$$

$$z = P^{\frac{1}{4}} = 16^{\frac{1}{4}}[(\cos(180 + 360n) + i \sin(180 + 360n))]^{\frac{1}{4}}$$

$$\text{For } n = 0$$

$$z = 2(\cos 45 + i \sin 45)$$

$$z = \sqrt{2} + \sqrt{2}i$$

$$\text{for } n = 1$$

$$z = 2(\cos 540 + i \sin 540)$$

$$z = 2(\cos 135 + i \sin 135)$$

$$= -\sqrt{2} + i\sqrt{2}$$

$$\text{For } n = 2, z = 2(\cos 225 + i \sin 225)$$

$$= -\sqrt{2} - i\sqrt{2}$$

$$\text{For } n = 3, z = 2(\cos 315 + i \sin 315)$$

$$z = \sqrt{2} - \sqrt{2}i$$

$$\Rightarrow \text{For } z = (-16 + 0i)^{\frac{1}{4}}$$

$$z = \sqrt{2} - (\sqrt{2})i, -\sqrt{2} + (\sqrt{2})i$$

$$\sqrt{2} + (\sqrt{2})i, -\sqrt{2} - (\sqrt{2})i$$

Example XIX

Find the cube roots of 27i

$$z = (0 + 27i)^{\frac{1}{3}}$$

$$\text{let } P = 0 + 27i$$

$$|P| = \sqrt{0^2 + 27^2} = 27$$

$$\arg P = \tan^{-1}\left(\frac{27}{0}\right) = 90$$

$$P = 27(\cos 90 + i \sin 90)$$

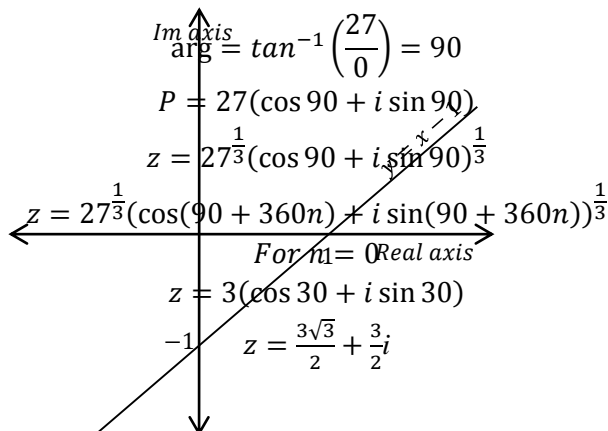
$$z = 27^{\frac{1}{3}}(\cos 90 + i \sin 90)^{\frac{1}{3}}$$

$$z = 27^{\frac{1}{3}}(\cos(90 + 360n) + i \sin(90 + 360n))^{\frac{1}{3}}$$

$$\text{For } n = 0$$

$$z = 3(\cos 30 + i \sin 30)$$

$$z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$



$$n = 1$$

$$z = 3(\cos 150 + i \sin 150)$$

$$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -\frac{3\sqrt{3}}{2} + \frac{3i}{2}$$

$$\text{for } n = 2$$

$$z = 3(\cos 270 + i \sin 270)$$

$$= -3i$$

Loci in the complex plane

What is a locus

A locus is a path possible position of a variable point, that obeys a given condition. It can be given as Cartesian equation or it can be described in words.

Example I

The complex number z is represented by the point P on the Argand diagram.

Given that $|z - 1 - i| = |z - 2|$ find in the simplest form the Cartesian equation of the locus

Solution

$$|z - 1 - i| = |z - 2|$$

$$\text{let } z = x + iy$$

$$|x + iy - 1 - i| = |x + iy - 2|$$

$$|x - 1 + (y - 1)i| = |x - 2 + iy|$$

$$\sqrt{(x - 1)^2 + (y - 1)^2} = \sqrt{(x - 2)^2 + y^2}$$

$$(x - 1)^2 + (y - 1)^2 = (x - 2)^2 + y^2$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2$$

$$-2x - 2y + 2 = -4x + 4$$

$$2x - 2 = 2y$$

$$y = x - 1$$

The locus is a straight line with a positive gradient ($y = x - 1$) which can be represented on the complex plane.

Example II

Given that $|z - 2| = 2|z + i|$. Show that the locus of P is a circle.

Solution

$$|z - 2| = 2|z + i|$$

Let $z = x + iy$

$$|x + iy - 2| = 2|x + iy + i|$$

$$|(x - 2) + iy| = 2|x + (y + 1)i|$$

$$\sqrt{(x - 2)^2 + y^2} = 2\sqrt{x^2 + (y + 1)^2}$$

$$(x - 2)^2 + y^2 = 4(x^2 + (y + 1)^2)$$

$$x^2 - 4x + 4 + y^2 = 4x^2 + 4y^2 + 8y + 4$$

$$0 = 3x^2 + 3y^2 + 4x + 8y$$

$$x^2 + y^2 + \frac{4}{3}x + \frac{8y}{3} = 0$$

This is sufficient to justify that locus is a circle.

Comparing $x^2 + y^2 + \frac{4}{3}x + \frac{8y}{3} = 0$ With

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = \frac{4}{3}$$

$$g = \frac{2}{3}$$

$$2fy = \frac{8y}{3}$$

$$f = \frac{4}{3}$$

$$\text{centre}(-g, -f)$$

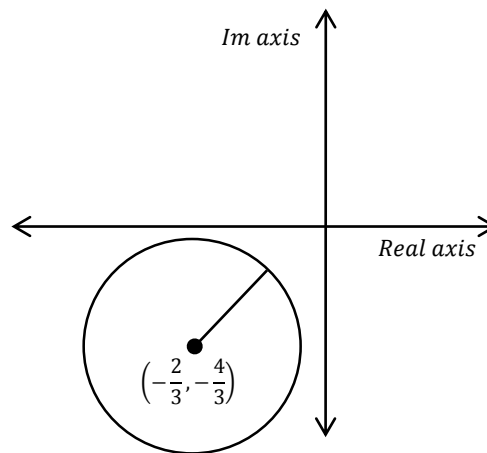
$$\text{centre}\left(-\frac{2}{3}, -\frac{4}{3}\right)$$

$$r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{\frac{4}{9} + \frac{16}{9} - 0}$$

$$r = \sqrt{\frac{20}{9}}$$

$$r = \frac{2}{3}\sqrt{5}$$

**Example III**

Show the region represented by $|z - 2 + i| < 1$

Solution

Let $z = x + iy$

$$|x + iy - 2 + i|$$

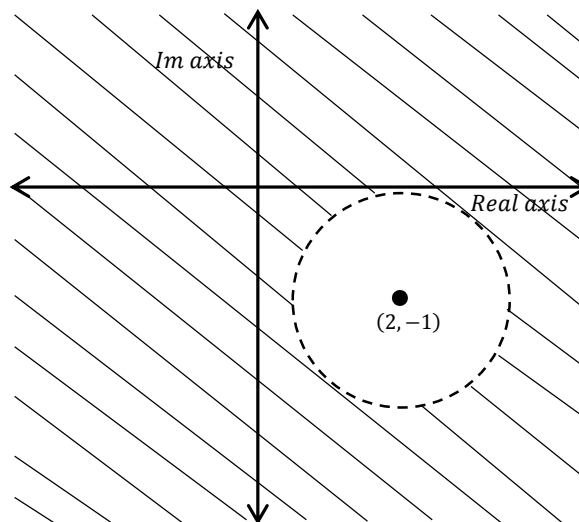
$$|x - 2 + (y + 1)i| < 1$$

$$\sqrt{(x - 2)^2 + (y + 1)^2} < 1$$

$$(x - 2)^2 + (y + 1)^2 < 1$$

It's a circle with centre (2, -1) and radius less than 1.

It can be illustrated on the argand diagram



In order to represent $(x - 2)^2 + (y + 1)^2 < 1$ on the diagram, we can either take a point inside the circle or outside the circle as our test point.

Taking (2,-1) as the test point.

$$\Rightarrow (2 - 2)^2 + (-1 + 1)^2 < 1$$

$$0 + 0 < 1$$

$$0 < 1$$

(2, -1)(the point inside the circle satisfies our locus).

It implies that (2,-1) lies in the wanted region.

Therefore, we shade the region outside the circle.

Example IV

Given that

$$\left| \frac{z-1}{z+1} \right| = 2$$

find the Cartesian equation of the locus of z and represent the locus by the sketch on the argand diagram. Shade the region for which the inequalities.

$$\left| \frac{z-1}{z+1} \right| > 2$$

Solution

$$z = x + iy$$

$$\left| \frac{x + iy - 1}{x + iy + 1} \right| = 2$$

$$\left| \frac{(x-1) + iy}{(x+1) + iy} \right| = 2$$

$$\frac{|x-1+iy|}{|(x+1)+iy|} = 2$$

$$|(x-1) + iy| = 2|(x+1) + iy|$$

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+1)^2 + y^2}$$

$$(x-1)^2 + y^2 = 4((x+1)^2 + y^2)$$

$$x^2 - 2x + 1 + y^2 = 4(x^2 + 2x + 1 + y^2)$$

$$3x^2 + 3y^2 + 10x + 3 = 0$$

$$x^2 + y^2 + \frac{10}{3}x + 1 = 0$$

The locus is a circle comparing

$$x^2 + y^2 + \frac{10}{3}x + 1 = 0 \text{ with}$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = \frac{10}{3}, \quad g = \frac{5}{3}, \quad 2f = 0 \text{ and } f = 0$$

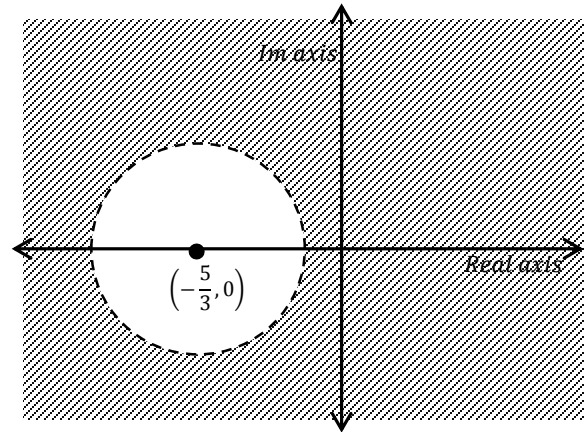
Center $(-\frac{5}{3}, 0)$

$$r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{\frac{25}{9} + 0 - 1} = \frac{4}{3}$$

$$\text{For } \left| \frac{z-1}{z+1} \right| > 2$$

$$\Rightarrow x^2 + y^2 + \frac{10}{3}x + 1 > 0$$



Example V

Shade the region represented by $|z - 1 - i| < 3$

Solution

Note: Shade the region represented by $|z - 1 - i| < 3$.

Implies that we shade the wanted region.

Let $x + iy$

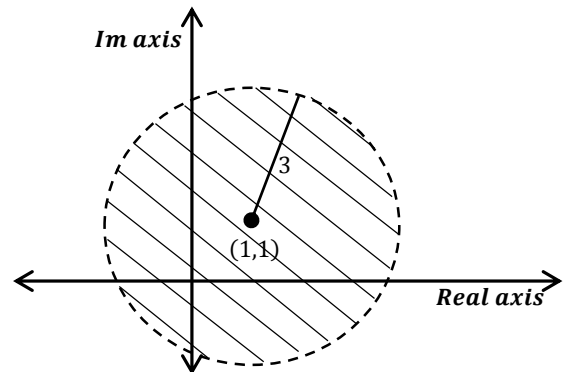
$$|x + iy - 1 - i| < 3$$

$$|x - 1 + i(y - 1)| < 3$$

$$\sqrt{(x-1)^2 + (y-1)^2} < 3$$

$$(x-1)^2 + (y-1)^2 < 9$$

It is a circle with centre (1, 1) and radius less than 9



Taking (1, 1) as our test point

$$(1-1)^2 + (1-1)^2$$

$$(0+0) < 9$$

\Rightarrow The region inside the circle is the wanted region.

Example VI

Show that when

$$\operatorname{Re} \left(\frac{z+i}{z+2} \right) = 0,$$

the point $P(x, y)$ lies on a circle with centre $(-1, -\frac{1}{2})$ and radius $\frac{1}{2}\sqrt{5}$

Solution

$$\operatorname{Re} \left(\frac{x+iy+i}{x+iy+2} \right) = 0$$

$$\operatorname{Re} \left(\frac{x+(y+1)i}{x+2+iy} \right) = 0$$

$$\operatorname{Re} \left[\frac{(x+(y+1)i)(x+2-iy)}{((x+2)+iy)(x+2-iy)} \right] = 0$$

$$\operatorname{Re} \left(\frac{x(x+2) - xyi + (y+1)(x+2)i + y(y+1)}{(x+2)^2 + y^2} \right)$$

$$\operatorname{Re} \left(\frac{x^2 + 2x + y^2 + y + [(y+1)(x+2) - xy]i}{(x+2)^2 + y^2} \right)$$

$$\operatorname{Re} \left(\frac{x^2 + 2x + y^2 + y + [(y+1)(x+2) - xy]i}{(x+2)^2 + y^2} \right) = 0$$

$$\Rightarrow \frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2} = 0$$

$$x^2 + y^2 + 2x + y = 0$$

Comparing with

$$x^2 + y^2 + 2x + y = 0 \text{ with}$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$2g = 2, g = 1$$

$$2fy = y$$

$$f = \frac{1}{2}$$

$$\text{centre} \left(-1, -\frac{1}{2} \right)$$

$$\text{radius} = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{1 + \frac{1}{4} - 0}$$

$$= \frac{\sqrt{5}}{2}$$

$$= \frac{1}{2}\sqrt{5}$$

Example VII

Given that $z = x + iy$, where x and y are real.

$$\operatorname{Im} \left(\frac{z+i}{z+2} \right) = 0$$

is equation of a straight line

Solution

$$\operatorname{Im} \left(\frac{x+iy+i}{x+iy+2} \right) = 0$$

$$\operatorname{Im} \left[\frac{(x+(y+1)i)(x+2-iy)}{(x+2+iy)(x+2-iy)} \right] = 0$$

$$\operatorname{Im} \left(\frac{x(x+2) - xyi + (y+1)(x+2)i + y(y+1)}{(x+2)^2 + y^2} \right)$$

$$= 0$$

$$\Rightarrow \frac{-xy + (y+1)(x+2)}{(x+2)^2 + y^2} = 0$$

$$\frac{-xy + xy + 2y + x + 2}{(x+2)^2 + y^2} = 0$$

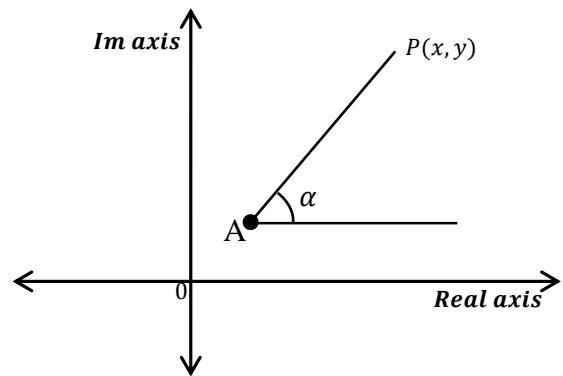
$$2y + x + 2 = 0$$

$$y = -\frac{x}{2} + 1$$

Which is a straight line with a negative gradient.

Loci in and diagram for arguments of complex numbers

If $\arg(z - A) = \alpha$ is the equation of half line with end point A inclined at an angle α to the real axis



Example I

Sketch the loci defined by the equation

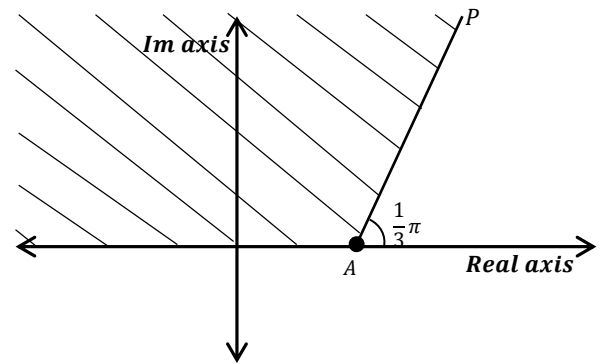
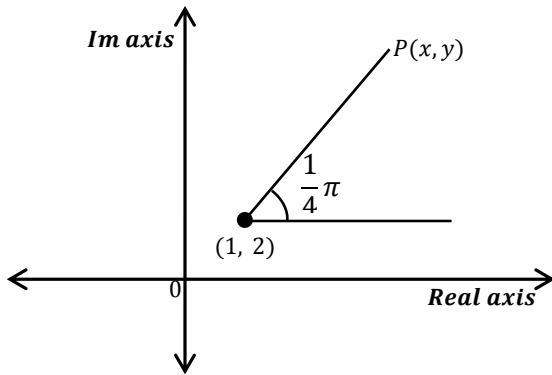
$$\arg(z - 1 - 2i) = \frac{1}{4}\pi$$

Solution

$$z - 1 - 2i = z - (1 + 2i)$$

Thus if A is a point representing $1 + 2i$

$\arg(z - (1 + 2i))$ is the angle AP makes with the positive real axis. Hence the equation $\arg(z - 1 - 2i) = \frac{1}{4}\pi$ represents the half line with end point $(1, 2)$ inclined at angle $\frac{1}{4}\pi$ to the real axis.



Example II

Sketch the locus of the equation.

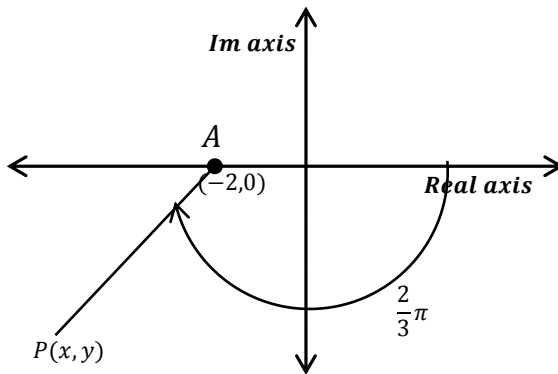
$$\arg(z + 2) = -\frac{2}{3}\pi$$

Solution

$$\arg(z + 2) = -\frac{2\pi}{3}$$

$$z + 2 = (z - -2)$$

Thus A is a point (-2, 0). Arg(z-2) is the angle AP makes with the real axis. Hence $\arg(z - -2) = -\frac{2}{3}\pi$ represents a half line with end point (-2, 0) inclined at angle $\frac{2}{3}\pi$ measured clockwise from the positive axis.



Example III

Show by shading the region represented by

$$\frac{1}{3}\pi \leq \arg(z - 2) \leq \pi$$

Solution

The equations $\arg(z - 2) = \frac{1}{3}\pi$ and $\arg(z - 2) = \pi$ represent half lines with end point (2, 0). Hence the inequality $\frac{1}{3}\pi \leq \arg(z - 2) \leq \pi$

Represent the two lines and region between them

Example IV

Sketch the separate argand diagram the loci defined by

$$(i) \arg(z + 1 - 3i) = -\frac{1}{6}\pi$$

$$(ii) \arg(z + 2 + i) = \frac{1}{2}\pi$$

Solution

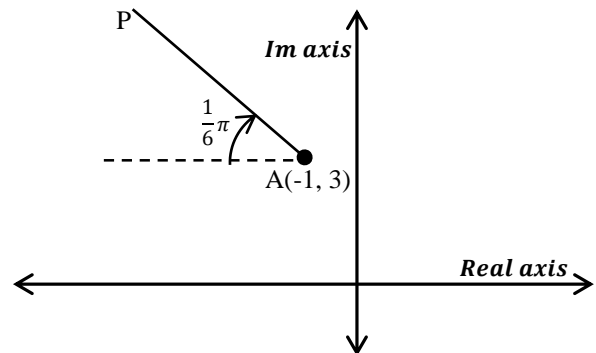
$$\arg(z + 1 - 3i) = -\frac{1}{6}\pi$$

$$z - (-1 + 3i) = -\frac{1}{6}\pi$$

Thus A is a point (-1, 3)

Arg(z - (-1 + 3i)) is the angle AP makes with the real axis Hence $\arg(z + 1 - 3i) = -\frac{1}{6}\pi$

is equation of the half line with end point (-1, 3) inclined at an angle of $\frac{1}{6}\pi$ measured clockwise from the real axis



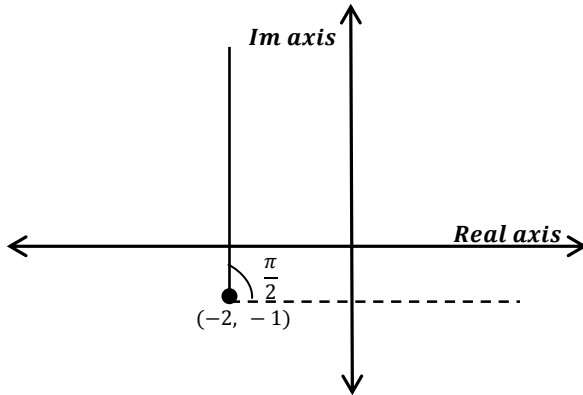
$$(ii) \arg(z + 2 + i) = \frac{1}{2}\pi$$

$$\arg(z - (-2 - i)) = \frac{1}{2}\pi$$

Thus, point A is (-2, -1).

$\arg(z - (-2 - i))$ is the angle AP makes with the real axis and $\arg(z + 2 + i) = \frac{1}{2}\pi$ is the equation of the

line through A inclined at an angle of $\frac{1}{2}\pi$ to the real axis



Sketching of loci involving $\arg\left(\frac{z-a}{z-b}\right) = \gamma$

Equation involving $\arg\left(\frac{z-a}{z-b}\right)$ are more difficult to interpret. If $\arg(z-a) = \alpha$,

$$\arg(z-b) = \beta, \arg\left(\frac{z-a}{z-b}\right) = \gamma,$$

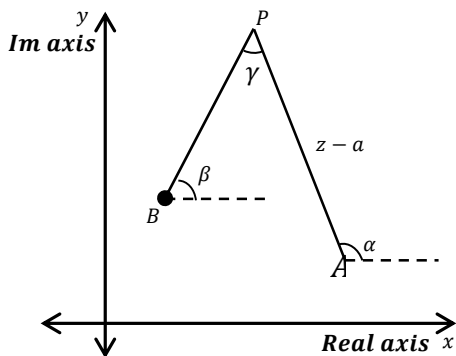
$$\arg(z-a) - \arg(z-b) = \gamma$$

$$\alpha - \beta = \gamma. \quad \gamma = (\alpha - \beta) \pm 2\pi \text{ if necessary}$$

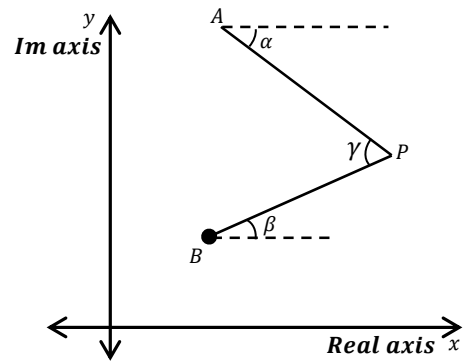
Thus γ is the angle which the vector AP makes with the vector BP.

If the turn from BP to AP is anti-clockwise the α is negative

$$\text{for } \arg\left(\frac{z-a}{z-b}\right) > 0$$



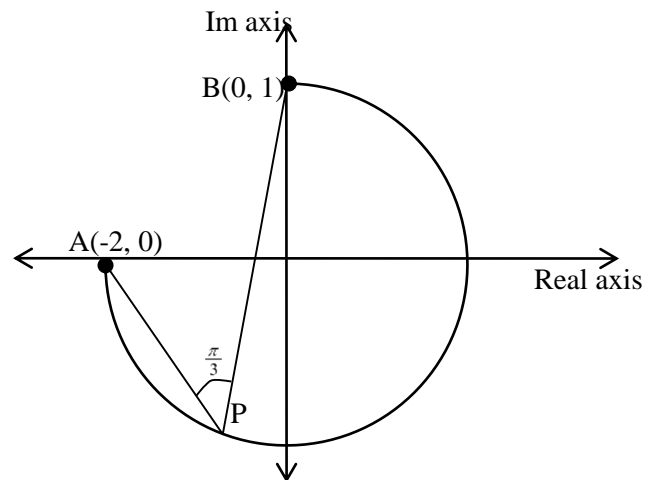
$$\arg\left(\frac{z-a}{z-b}\right) < 0$$



For instance, if $\arg\left(\frac{z-3}{z-1}\right) = \frac{1}{4}\pi$, then the locus of P is a circular arc with end point A(3, 0) and (1, 0) such that $\angle APB = \frac{1}{4}\pi$

Similarly if $\arg\left(\frac{z+2}{z-i}\right) = \frac{1}{3}\pi$ then the locus of P is a circular arc with end points A(-2, 0) and B(0, +1) such that $\angle APB = \frac{1}{3}\pi$ since both cases the given arguments are positive, the arcs must be drawn so that the turn from BP to AP is anti-clockwise.

$$\arg\left(\frac{z+2}{z-i}\right) = \frac{1}{3}\pi$$



Example II

Sketch on different argand diagram the loci defined by the equations.

$$(a) \arg\left(\frac{z-1}{z+1}\right) = \frac{1}{3}\pi$$

$$(b) \arg\left(\frac{z-3}{z-2i}\right) = \frac{1}{4}\pi$$

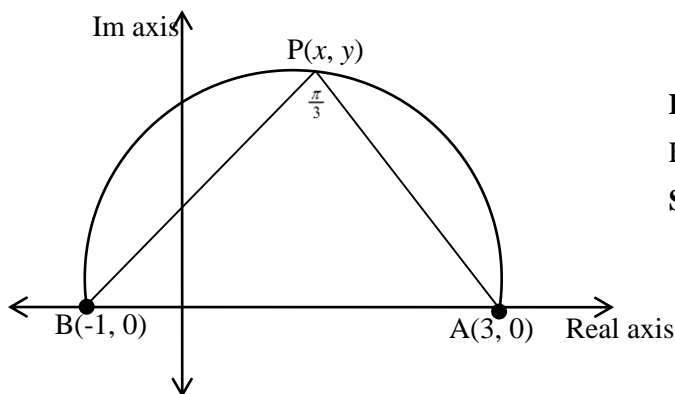
$$(c) \arg\left(\frac{z}{z-4+2i}\right) = \frac{1}{2}\pi$$

Solution

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{1}{3}\pi$$

The locus of P is a circular arc with end point A(1, 0) and B(-1, 0) such that

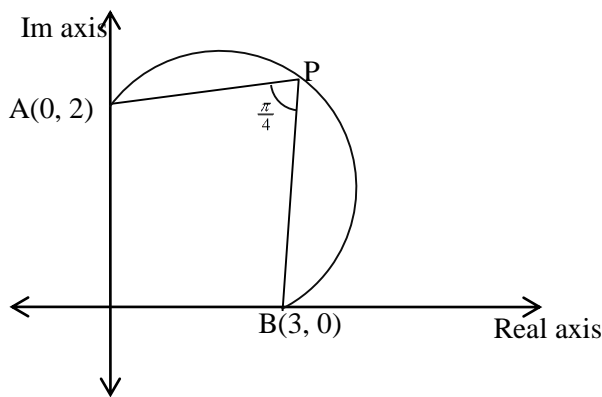
$$\angle APB = \frac{1}{3}\pi$$



(b) $\arg\left(\frac{z-3}{z-2i}\right) = \frac{1}{4}\pi$

The locus of P is a circular arc with end points (3, 0)

(0, 2) such that $\angle APB = \frac{1}{4}\pi$

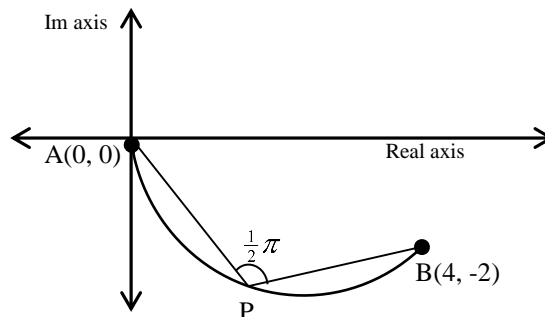


(c) $\arg\left(\frac{z}{z-4+2i}\right) = \frac{1}{2}\pi$

$$\arg\left(\frac{z}{z-(4-2i)}\right) = \frac{1}{2}\pi$$

$\arg\left(\frac{z}{z-4+2i}\right)$ is a circle with end points

A(0, 0) and B(4, -2) such that $\angle APB = \frac{1}{2}\pi$



Example

Find the locus of $\arg\left(\frac{z}{z-6}\right) = \frac{\pi}{2}$

Solution

let $z = x + iy$

$$\arg\left(\frac{z}{z-6}\right) = \arg z - \arg(z-6)$$

$$\Rightarrow \arg(z) - \arg(z-6) = \frac{\pi}{2}$$

$$\arg(x + iy) - \arg(x + iy - 6) = \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x-6}\right) = \frac{\pi}{2}$$

let $A = \tan^{-1}\left(\frac{y}{x}\right)$

$$\tan A = \frac{y}{x}$$

$$B = \tan^{-1}\left(\frac{y}{x-6}\right)$$

$$\tan B = \frac{y}{x-6}$$

$$(A - B) = \frac{\pi}{2}$$

$$\tan(A - B) = \tan\left(\frac{\pi}{2}\right)$$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \infty$$

$$\frac{\frac{y}{x} - \frac{y}{x-6}}{1 + \frac{y}{x} \cdot \frac{y}{x-6}} = \infty$$

$$1 + \frac{y^2}{x(x-6)}$$

$$\frac{y(x-6) - xy}{x(x-6)} = \infty$$

$$\frac{x^2 - 6x + y^2}{x(x-6)} = \infty$$

$$\frac{xy - 6y - xy}{x^2 + y^2 - 6x} = \infty$$

$\Rightarrow x^2 + y^2 - 6x = 0$ which is a circle.

Revision Exercise 1

1. Prove that if $|z| = r$, then $zz^* = r^2$.

2. Express $\sqrt{3} + i$ in modulus-argument form. Hence find $(\sqrt{3} + i)^{10}$ and $\frac{1}{(\sqrt{3} + i)^7}$ in the form $a + ib$.
3. Express $-1 + i$ in modulus-argument form. Hence show that $(-1 + i)^{16}$ is real and that $\frac{1}{(-1 + i)^6}$ is purely imaginary, giving the value of each.
4. Simplify the following expression:
 (a) $\frac{(\cos \frac{2\pi}{7} - i \sin \frac{2\pi}{7})^3}{(\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7})^4}$ (b) $\frac{(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})^8}{(\cos \frac{3\pi}{5} - i \sin \frac{3\pi}{5})^3}$
5. Find the expressions for $\cos 3\theta$ in terms of $\cos \theta$, $\sin 3\theta$ in terms of $\sin \theta$ and $\tan 3\theta$ in terms of $\tan \theta$.
6. Express $\sin 5\theta$ and $\cos 5\theta / \cos \theta$ in terms of $\sin \theta$.
7. Prove that $\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$. By considering the equation $\tan 5\theta = 0$, show that $\tan^2(\pi/5) = 5 - 2\sqrt{5}$.
8. Find expressions for $\cos 6\theta / \sin \theta$ in terms of $\cos \theta$ and for $\tan 6\theta$ in terms of $\tan \theta$.
9. Express in terms of cosines of multiples of θ :
 (a) $\cos^5 \theta$ (b) $\cos^7 \theta$ (c) $\cos^4 \theta$
10. Express in terms of sines of multiples of θ :
 (a) $\sin^3 \theta$ (b) $\sin^7 \theta$ (c) $\cos^4 \theta \sin^3 \theta$
11. Prove that $\cos^6 \theta + \sin^6 \theta = \frac{1}{8}(3 \cos^4 \theta + 5)$
12. Evaluate (a) $\int_0^{\pi} \sin^4 \theta \, d\theta$ (b) $\int_0^{\frac{\pi}{2}} \cos^4 \theta \sin^2 \theta \, d\theta$
13. (a) Express the following complex numbers in a form having a real denominator.
 $\frac{1}{3 - 2i}$, $\frac{1}{(1 + i)^2}$
 (b) Find the modulus and principal arguments of each of the complex numbers $Z = 1 + 2i$ and $W = 2 - i$, and represent Z and W clearly by points A and B in an Argand diagram. Find also the sum and product of Z and W and mark the corresponding points C and D in your diagram.
14. If the complex number $x + iy$ is denoted by Z , then the complex conjugate number $x - iy$ is denoted by Z^* ,
 (a) Express $|Z^*|$ and $\arg(Z^*)$ in terms of $|Z|$ and $\arg(Z)$.
- (b) If a , b , and c are real numbers, prove that if $aZ^2 + bZ + c = 0$, then $a(Z^*)^2 + b(Z^*) + c = 0$
- (c) If p and q are complex numbers and $q \neq 0$, prove $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$
15. Find the values of a and b such that $(a + ib)^2 = i$. Hence or otherwise solve the equation $z^2 + 2z + 1 - i = 0$, giving your answer in the form $p + iq$, where p and q are real numbers.
16. If $Z = \frac{1}{2}(1 + i)$, write down the modulus and argument for each of the numbers Z , Z^2 , Z^3 , Z^4 . Hence or otherwise, show in the Argand diagram, the points representing the number $1 + Z + Z^2 + Z^3 + Z^4$.
17. If $Z = 3 - 4i$, find
 (i) Z^* (ii) ZZ^* (iii) $(ZZ^*)^*$
18. Simplify each of the following:
 (a) $(3 + 4i) + (2 + 3i)$ (b) $(2 - 4i) - 3(5 - 3i)$
 (c) $(2i)^2$ (d) i^4
19. Simplify each of the following:
 (a) $(2 + i)(3 - i)$ (b) $(5 - 2i)(6 + i)$
 (c) $(4 - 3i)(1 - i)$ (d) $(3 + i)(2 - 5i)$
20. Express each of the following in the form $a + ib$
 (a) $\frac{20}{3 + i}$ (b) $\frac{4}{1 + i}$
 (c) $\frac{2i}{1 - i}$ (d) $\frac{1}{1 - 2i}$
21. Solve the following equations:
 (a) $x^2 + 25 = 0$
 (b) $2x^2 + 32 = 0$
 (c) $4x^2 + 9 = 0$
 (d) $x^2 + 2x + 5 = 0$
22. If $3 - 2i$ and $1 + i$ are two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + c = 0$, find the values of a , b , c , d and e .
23. Find the square roots of the following complex numbers:
 (a) $5 + 2i$
 (b) $15 + 8i$
 (c) $7 - 24i$
24. Find the quadratic equations have the roots:
 (a) $3i, -3i$ (b) $1 + 2i, 1 - 2i$
 (c) $2 + i, 2 - i$ (d) $2 + 3i, 2 - 3i$
25. Find real and imaginary parts of the complex Z when:
 (i) $\frac{Z}{Z + 1} = 1 + 2i$
 (ii) $\frac{Z + i}{Z + 1} = \frac{Z + i}{Z - 3}$
26. Find the modulus and principal argument of the following complex numbers
 (a) $3i$ (b) 15 (c) $-3i$ (d) -1

27. Find the modulus and principle argument of:
- (a) $\frac{1-i}{1+i}$ (b) $\frac{-1-7i}{4+3i}$
- (c) $\frac{1+i}{2-i}$ (d) $\frac{(3+i)^2}{1-i}$
28. If Z_1 and Z_2 are complex numbers, solve the simultaneous equations
- $$4Z^1 + 3Z^2 = 23$$
- $$Z^1 + iZ^2 = 6$$
- giving your answer in the form $x + iy$
29. Given that $2 + i$ is a root of the equation $Z^3 - 11Z + 20 = 0$. Find the remaining roots.
30. Show that $1 + i$ is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down the quadratic factor of $x^4 + 3x^2 - 6x + 10$ and find all the roots of the equation.
31. The complex number satisfies the equation $\frac{Z}{Z+2} = 2 - i$. Find the real and imaginary parts of Z and the modulus and argument of Z .
32. If $Z_1 = 4(\cos \frac{13\pi}{24} + i \sin \frac{13\pi}{24})$ and $Z_2 = 2(\cos \frac{5\pi}{24} + i \sin \frac{5\pi}{24})$, find $\frac{Z_1}{Z_2}$ and $Z_1 Z_2$ in the form $a + ib$.
33. If $Z_1 = 2\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ and $Z_2 = 6(\cos \frac{-3\pi}{4} + i \sin \frac{-3\pi}{4})$, find:
- (i) $\left| \frac{Z_1}{Z_2} \right|$ (ii) $\arg\left(\frac{Z_1}{Z_2}\right)$ (iii) $\left| \frac{Z_2}{Z_1} \right|$
- (iv) $\arg\left(\frac{Z_2}{Z_1}\right)$
34. One root of the equation $Z^2 + aZ + b = 0$ where a and b are real constants, is $2 + 3i$. Find the values of a and b .
35. If Z_1 and Z_2 are two complex numbers such that $|Z_1 - Z_2| = Z_1 + Z_2$, show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$
36. (a) Find the modulus and argument of $\frac{(2-i)^2(3i-1)}{i+3}$
- (b) If $Z_1 = \frac{1+7i}{1-i}$ and $Z_2 = \frac{17-7i}{2+2i}$. Find the moduli of Z_1 , Z_2 , $Z_1 + Z_2$ and $Z_1 Z_2$.
37. Use Demoiivre's theorem to show that:
- $$\frac{(\cos 3\theta + i \sin 3\theta)^5 (\cos \theta - i \sin \theta)^3}{(\cos 5\theta + i \sin 5\theta)^7 (\cos 2\theta - i \sin 2\theta)^5} = \cos 13\theta - i \sin 13\theta$$
38. Use Demoiivre's theorem to show that:
- $$\cos 4\theta = \cos^4 \theta - 6\cos^2 \theta \sin^2 \theta + \sin^4 \theta$$
- $$\sin 4\theta = 4\cos^3 \theta \sin \theta - 4\cos \theta \sin^3 \theta$$
39. Show that $\left(\frac{1+\sin \theta + i \cos \theta}{1+\sin \theta - i \cos \theta}\right)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i \sin n\left(\frac{\pi}{2} - \theta\right)$
40. Use Demoiivre's theorem to find the value of $\frac{\sqrt{3}-1}{\sqrt{3}+1}$
41. Find the two square roots of I and the four values of $(-16)^{1/4}$.
42. Find the three roots of the equation $(1 - Z)^3 = Z^3$
43. If W is a complex cube root of unity, show that $(1 + W - W^2)^3 - (1 - W + W^2)^3 = 0$
44. Use Demoiivre's theorem to find the four fourth roots of $8(-1 + i\sqrt{3})$ in the form $a + ib$, giving a and b correct to 2 decimal places.
45. Use Demoiivre's theorem to show that $\frac{\cos 5x}{\cos x} = 1 - 12\sin^2 x + 16\sin^4 x$
46. Prove that if $\frac{Z-6i}{Z+8}$ is real, the locus of the point representing the complex number Z in the Argand diagram is a straight line.
47. Prove that if $\frac{Z-2i}{2Z-1}$ is purely imaginary, the locus of the point representing Z in the Argand diagram is a circle and find its radius.
48. If Z is a complex number and $\left|\frac{Z-i}{Z+1}\right| = 2$, find the equation of the curve in the Argand diagram on which the point representing it lie.
49. The complex numbers $Z - 2$ and $Z - 2i$ have arguments which are
- (i) equal and
- (ii) differ by $\frac{1}{2}\pi$ and each argument lies between $-\pi$ and π . In each case, find the locus of the point which represents Z in the Argand diagram and illustrate by a sketch.
50. Show by shading on an Argand diagram the region in which both $|Z - 3 - i| \geq |Z - 3 - 5i|$

Answers

- 1.
2. (a) 1, (b) $-i$ (c) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (d) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3. $2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$; $512 - 512\sqrt{3}i$, $\frac{\sqrt{3}}{256} + \frac{1}{256}i$
4. $\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$; $256 - \frac{1}{8}i$
5. (a) 1, (b) -1
6. $4\cos^3 \theta - 3\cos \theta - 4\sin^3 \theta$, $\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$
7. $16\sin^5 \theta - 20\sin^3 \theta + 5\sin \theta$, $1 - 12\sin^2 \theta + 16\sin^4 \theta$
8. .

9. $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1, 32\cos^5\theta - 32\cos^3\theta + 6\cos = \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta}$
10. (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos \theta)$,
 (b) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos \theta)$
 (c) $\frac{1}{16}(2\cos \theta - \cos 3\theta - \cos 5\theta)$
11. (a) $\frac{1}{4}(3\sin \theta - \sin 3\theta)$,
 $\frac{1}{64}(35\sin \theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$,
 (c) $\frac{1}{64}(3\sin \theta + \sin 3\theta - \sin 5\theta - \sin 7\theta)$
12. (a) $\frac{3\pi}{8}$, (b) $\frac{\pi}{32}$
13. (a) $\frac{3+2i}{13}, \frac{1}{2}i$ (b) $\sqrt{5}, 63.4^\circ, \sqrt{5}, -26.6^\circ, 3+i, 4+3i$.
15. $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}$ or $a = \frac{-1}{\sqrt{2}}, b = \frac{-1}{\sqrt{2}}$
 $Z = -1 + \frac{1}{\sqrt{2} + \frac{i}{\sqrt{2}}}$ or $Z = -1 - \frac{1}{\sqrt{2} - \frac{i}{\sqrt{2}}}$
16. $\frac{\sqrt{2}}{2}, 45^\circ; \frac{1}{2}, 90^\circ; \frac{\sqrt{2}}{4}, 135^\circ; \frac{1}{4}, 180^\circ$
17. (i) $3 + 4i$ (ii) 25 (iii) $-7 + 24i$
18. (a) $5 + 7i$ (b) $-13 + 5i$ (c) -4 (d) 1
19. (a) $7 + i$ (b) $32 - 7i$ (c) $1 - 7i$ (d) $11 - 13i$
20. (a) $6 - 2i$ (b) $2 - 2i$ (c) $-1 + i$ (d) $\frac{1}{5} + \frac{2}{5}i$
21. (a) $x \pm 5i$ (b) $x = \pm 4i$ (c) $\pm \frac{3}{2}i$ (d) $x = -1 \pm 2i$
22. $a = 1, b = -8, c = 27, d = -38, e = 26$
23. (a) $\pm(3 + 2i)$ (b) $\pm(4 + i)$ (c) $\pm(4 - 3i)$
24. (a) $x^2 + 9 = 0$ (b) $x^2 - 2x + 5 = 0$
 (c) $x^2 - 4x + 5 = 0$ (d) $x^2 - 4x + 13 = 0$
25. (i) $-1, \frac{1}{2}$ (ii) $\frac{1}{5}, \frac{-2}{5}$
26. (a) $3, \pi/2$ (b) $15, 0$ (c) $3, -\pi/2$ (d) $1, \pi$
27. (a) $1, -\pi/2$ (b) $\sqrt{2}, \frac{-3\pi}{4}$, (c) $\frac{\sqrt{10}}{5}, 1.25$
28. $2 + 3i$ 19. $2 - i, -4$
31. (i) $\text{Re}(Z) = -3, \text{Im}(Z) = -1$ (ii) $\sqrt{10}, -2.82$ rads
32. $1 + \sqrt{3}i; -4\sqrt{2} + 4\sqrt{2}i$
33. (i) $1/3$ (ii) $\frac{-7\pi}{12}$ (iii) 3 (iv) $\frac{7\pi}{12}$
34. $-4, 13$
36. (a) $5, 0.6435$ rad (b) $5, 6.5, 2.061, 32.5$
41. $\frac{\pm(1+i)}{\sqrt{2}}, \pm\sqrt{2} \pm i\sqrt{2}$
42. $\frac{1}{2}, \frac{1}{2}(1 \pm i\sqrt{3})$. 44. $\pm(1.73 + i), \pm(1 - 1.73i)$
47. centre $\frac{1}{4} + i$, radius $\frac{1}{4}\sqrt{7}$
48. $\left(x + \frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$
49. (i) $x + y = 2$ (ii) $(x - 1)^2 + (y - 1)^2 = 2$

Exercise 2

Show on the Argand diagram the region represented by the following:

- $\arg z = \frac{1}{4}\pi$,
- $\arg(z - i) = \frac{1}{3}\pi$
- $\arg(z + 1 - 3i) = \frac{1}{6}\pi$
- $\arg(z - 3 + 2i) = \pi$
- $\arg(z + 2 + i) = \frac{1}{2}\pi$
- $\arg(z - 1 - i) = -\frac{1}{4}\pi$
- $|z + 1| = |z - 3|$,
- $|z| = |z - 6i|$
- $\left|\frac{z-i}{z-1}\right| = 1$
- (a) $\arg\left(\frac{z-1}{z+1}\right) = \frac{1}{3}\pi$
- (a) $\arg\left(\frac{z-3}{z-2i}\right) = \frac{1}{4}\pi$ (b) $\left(\frac{z}{z-4+2i}\right) = \frac{1}{2}\pi$

In questions 12 to 24 find the Cartesian equation of the locus of the point P representing the complex number z . Sketch the locus of P each case.

- $2|z + 1| = |z - 2|$
- $|z + 4i| = 3|z - 4|$
- $\left|\frac{z}{z-4}\right| = 5$
- $\left|\frac{z+i}{z-5-2i}\right| = 1$
- $\left|\frac{z}{z+6}\right| = 5$
- $\left|\frac{z-1}{z+1-i}\right| = \frac{2}{3}$
- $z - 5 = \lambda i(z + 5)$, where λ is a real parameter
- $\frac{z+2i}{z-2} = \lambda i$, where λ is a real number.
- $z = 3i + \lambda(2 + 5i)$, where λ is a real parameter.
- $\text{Im}(z^2) = 2$
- $\text{Re}(z^2) = 1$
- $\text{Re}\left(z - \frac{1}{z}\right) = 0$
- $\text{Im}\left(z + \frac{9}{z}\right) = 0$

In questions 27 to 34 shade in separate Argand diagrams the regions represented by:

- $|z - i| \leq 3$
- $|z - 4 + 3i| < 4$
- $0 \leq \arg z \leq \frac{1}{3}\pi$
- $\frac{1}{4}\pi < \arg z < \frac{3}{4}\pi$

29. $-\frac{1}{6}\pi < \arg(z-1) < \frac{1}{6}\pi$
 30. $-\frac{1}{2}\pi \leq \arg(z+i) \leq \frac{2}{3}\pi$
 31. $|z| > |z+2|$
 32. $|z+i| \leq |z-3i|$
 33. Represent each of the following loci in an Argand diagram.
 (a) $\arg(z-1) = \arg(z+1)$
 (b) $\arg z = \arg(z-1-i)$
 (c) $\arg(z-2) = \pi + \arg z$
 (d) $\arg(z-1) = \pi + \arg(z-i)$
 34. Find the least value of $|z+4|$ for which
 (a) $\operatorname{Re}(z) = 5$ (b) $\operatorname{Im}(z) = 3$
 (c) $|z| = 1$ (d) $\arg z = \frac{1}{4}\pi$
 35. Given that the complex number z varies such that $|z-7| = 3$, find the greatest and least values of $|z-i|$.
 36. Given that the complex number w and z vary subject to the conditions $|z-12| = 7$ and $|z-i| = 4$, find the greatest and least values of $|w-z|$.
 37. In an Argand diagram, the point P represents the complex number z , where $z = x + iy$. Given that $z+2 = \lambda i(z+8)$, where λ is a real parameter, find the Cartesian equation of the locus of P as λ varies. If also $z = \mu(4+3i)$, where μ is real, prove that there is only one possible position for P .
 38. (i) Represent on the same Argand diagram the loci given by the equations $|z-3| = 3$ and $|z| = |z-2|$. Obtain the complex numbers corresponding to the point of intersection of these loci. (ii) Find a complex number z whose argument is $\pi/4$ and which satisfies the equation $|z+2+i| = |z-4+i|$.

Answers

12. $x^2 + y^2 + 4x = 0$, 13. $x^2 + y^2 - 9x - 9y + 16$
 15. $5x + 3y = 14$. 16. $2x^2 + 2y^2 + 25x + 75 = 0$
 17. $5x^2 + 5y^2 - 26x + 8y + 1 = 0$.
 18. $x^2 + y^2 = 25$, excluding $(-5, 0)$
 19. $x^2 + y^2 - 2x + 2y = 0$, excluding $(2, 0)$.
 20. $5x - 2y + 6 = 0$ 21. $xy = 1$. 22. $x^2 - y^2 = 1$
 23. $x(x^2 + y^2 - 1) = 0$, excluding $(0, 0)$
 24. $y(x^2 + y^2 - 9) = 0$, excluding $(0, 0)$.
 34. (a) 9, (b) 3, (c) 3, (d) 4.
 35. $5\sqrt{2} + 3$, $5\sqrt{2} - 3$. 38. 24, 2.
 37. $x^2 + y^2 + 10x + 16 = 0$
 38. (i) $1 \pm i\sqrt{5}$ (ii) $1 + i$.

Revision Exercise 3

Show on the Argand diagram the region represented by the following:

1. $\left(\frac{z+1}{z-1}\right) = \frac{1}{3}\pi$
 2. $\left|\frac{z-2-3i}{z+2+i}\right| = 1$
 3. Express the complex number $z_1 = \frac{11+2i}{3-4i}$ in the form $x + iy$ where x and y are real. Given that $z_2 = 2 - 5i$, find the distance between the points in the Argand diagram which represent z_1 and z_2 . Determine the real numbers α and β such that $\alpha z_1 + \beta z_2 = -4 + i$.
 4. (i) Find two complex numbers z satisfying the equation $z^2 = -8 - 6i$.
 (ii) Solve the equation $z^2 - (3-i)z + 4 = 0$ and represent the solutions on an Argand diagram by vectors \overline{OA} and \overline{OB} , where O is the origin. Show that triangle OAB is right-angled.
 5. If z and w are complex numbers, show that:
 $|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$
 Interpret your results geometrically.
 6. A regular octagon is inscribed in the circle $|z| = 1$ in the complex plane and one of its vertices represents the number $\frac{1}{\sqrt{2}}(1+i)$. Find the numbers represented by the other vertices.
 7. (i) Two complex numbers z_1 and z_2 each have arguments between 0 and π . If $z_1 z_2 = i - \sqrt{3}$ and $\frac{z_1}{z_2} = 2i$, find the values of z_1 and z_2 giving the modulus and argument of each.
 (ii) Obtain in the form $a + ib$ the solutions of the equation $z^2 - 2z + 5 = 0$, and represent the solutions on an Argand diagram by the points A and B .
 The equation $z^2 - 2pz + q = 0$ is such that p and q are real, and its solutions in the Argand diagram are represented by the points C and D . Find in the simplest form the algebraic relation satisfied by p and q in each of the following cases:
 (a) $p^2 < q$, $p \neq 1$ and A, B, C, D are the vertices of a triangle;
 (b) $p^2 > q$ and $\angle CAD = \frac{1}{2}\pi$
 8. (a) If $-\pi < \arg z_1 + \arg z_2 \leq \pi$, show that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. The complex numbers $a = 4\sqrt{3} + 2i$ and $b = \sqrt{3} + 7i$ are represented in the Argand diagram by points A and B respectively. O is the origin. Show that triangle OAB is equilateral and find the complex number c which the point C represents where $OABC$ is a rhombus. Calculate $|c|$ and $\arg c$.

- (b) z is a complex number such that $z = \frac{p}{2-q} + \frac{q}{1+3i}$ where p and q are real. If $\arg z = \pi/2$ and $|z| = 7$ find the values of p and q .
9. .
10. (a) Show that $(1 + 3i)^3 = -(26 + 18i)$.
 (b) Find the three roots z_1, z_2, z_3 of the equation $z^3 = -1$
 (c) Find in the form $a + ib$, the three roots z'_1, z'_2, z'_3 of the equation $z^3 = 26 + 18i$.
 (d) Indicate in the same Argand diagram the points represented by z_r and z'_r for $r = 1, 2, 3$, and prove that the roots of the equations may be paired so that $|z_1 - z_2| = z_2 - z'_2 = |z_3 - z'_3| = 3$.
11. Write down or obtain the non-real cube roots of unity, w_1 and w_2 , in the form $a + ib$, where a and b are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent w_1 and w_2 , respectively and centre of the circumscribing circle of the hexagon is the point $(1, 0)$. Determine in the form $a + ib$, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers.
12. A complex number w is such that $w^3 = 1$ and $w \neq 1$. Show that:
 (i) $w^2 + w + 1 = 0$
 (ii) $(x + a + b)(x + wa + w^2b)(x + w^2a + wb)$ is real for real x, a and b , and simplify this product. Hence or otherwise find the three roots of the equation $x^3 - 6x + 6 = 0$, giving your answers in terms of w and cube roots of integers.
13. (i) Find, without the use of tables, the two square roots of $5 - 12i$ in the form $x + iy$, where x and y are real.
 (ii) Represent on an Argand diagram the loci $|z - 2| = 2$ and $|z - 4| = 7$. Calculate the complex numbers corresponding to the points of intersection of these loci.
14. (i) Given that $(1 + 5i)p - 2q = 7i$, find p and q when (a) p and q are real (b) p and q are conjugate complex numbers.
 (ii) Shade on the Argand diagram the region for which $3\pi/4 < \arg z < \pi$ and $0 < |z| < 1$. Choose a point in the region and label it A . If A represents the complex number z , label clearly the points B, C, D and E which represent $-z, iz, z + 1$ and z^2 respectively.
15. (i) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Find the other roots of the equation.
 (ii) Sketch the curve in the Argand diagram defined by $|z - 1| = 1, \text{Im } z \geq 0$. Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the value of $\arg z$ and $\arg(z - 2)$ at P .
16. (i) If $z = 1 + i\sqrt{3}$, find $|z|$ and $|z^5|$, and also the values of $\arg z$ and $\arg(z^5)$ lying between $-\pi$ and π . Show that $\text{Re}(z^5) = 16$ and find the value of $\text{Im}(z^5)$.
 (ii) Draw the line $|z| = |z - 4|$ and the half line $\arg(z - i) = \pi/4$ in the Argand diagram. Hence find the complex number that satisfies both equations.
17. (i) Without using tables, simplify $\frac{(\cos \frac{\pi}{9} + i \cos \frac{\pi}{9})^4}{(\cos \frac{\pi}{9} - i \sin \frac{\pi}{9})^5}$.
 (ii) Express $z_1 = \frac{7+4i}{3-2i}$ in the form $p + qi$, where p and q are real. Sketch in an Argand diagram the locus of the points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of z subject to this condition.
18. (i) Given that $z = 1 - i$, find the values of $r (> 0)$ and $\theta, -\pi < \theta < \pi$, such that $z = r(\cos \theta + i \sin \theta)$. Hence or otherwise find $1/z$ and z^6 , expressing your answers in the form $p + iq$, where $q, r \in \mathbb{R}$.
 (ii) Sketch on an Argand diagram the set of points corresponding to the set A , where $A = \{z: z \in \mathbb{C}, \arg(z - i) = \pi/4\}$. Show that the set of points corresponding to the set B , where $B = \{z: z \in \mathbb{C}, |z + 7i| = 2|z - 1|\}$, forms a circle in the Argand diagram. If the centre of this circle represents the numbers z_1 , show that $z_1 \in A$.
19. Use De Moivre's theorem to show that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$
20. (i) If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real. Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$ giving the coordinates of its centre.
 (ii) If $z = \cos \theta + i \sin \theta$, show that:

$$z = \frac{1}{z} = 2i \sin \theta \quad z^n = \frac{1}{z^n} = 2i \sin n\theta$$
Hence or otherwise, show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin \theta$
21. .
22. (i) Given that x and y are real, find the values of x and y which make satisfy the equation $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$

- (ii) Given that $z = x + iy$, where x and y are real,
 (a) Show that $\text{Im}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a straight line (b) Show that, when $\text{Re}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a circle with centre $(-1, -\frac{1}{2})$ and radius $\frac{1}{2}\sqrt{5}$
23. (i) Find $|z|$ and $\arg z$ for which the complex numbers z given by (a) $12 - 5i$, (b) $\frac{1+2i}{2-i}$, giving the argument in degrees (to the nearest degree) such that $-180^\circ < \arg z \leq 180^\circ$.
 (ii) By expressing $\sqrt{3} - i$ in modulus-argument form, or otherwise, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.
 (iii) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at $(-1, -1)$. Write down in modulus form the condition satisfied by the complex number z represented by point P .
24. Sketch the circle C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P representing the non-zero complex number z lies on C . Express $|z|$ in terms of θ , the argument of z . Given that $z' = 1/z$, find the modulus and argument of z' in terms of θ . Show that, whatever the position of P on the circle C , the point P' representing z' lies on a certain line, the equation of which is to be determined.
25. (a) The sum of the infinite series $1 + z + z^2 + z^3 + \dots$ for values of z such that $|z| < 1$ is $1/(1 - z)$. By substituting $z = \frac{1}{2}(\cos \theta + i \sin \theta)$ in this result and using De Moivre's theorem, or otherwise, prove that
- $$\frac{1}{2} \sin \theta + \frac{1}{2^2} \sin 2\theta + \frac{1}{2^n} \sin n\theta + \dots = \frac{2 \sin \theta}{5 - 4 \cos \theta}$$

- 10.(b) $-1, \frac{1}{2} \pm \frac{3}{2}i$, (c) $-1 - 3i$,
 $\frac{1}{2}(1 - 3\sqrt{3}) + \frac{1}{2}(3 + \sqrt{3})i$
 11. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i; 1 \pm i\sqrt{3}; \frac{5}{2} \pm \frac{\sqrt{3}}{2}i; 28$
 12. (ii) $x^3 - 3abx + a^3 + b^3; \sqrt[3]{2} - \sqrt[3]{4}, \omega\sqrt[3]{2} - \omega^2\sqrt[3]{4}, \omega^2\sqrt[3]{2} - \omega\sqrt[3]{4}$
 13. (i) $\pm(3 - 2i)$, (ii) $3 \pm i\sqrt{3}$.
 14. (i) (a) $7/5, -4/5$; (b) $2 \pm i$
 15. (i) $1 - i, -1 \pm 2i$, (ii) $\frac{1}{2}(3 + i\sqrt{3})$; $\pi/6, 2\pi/3$
 16. (i) $2, 32, \pi/3, -\pi/3; -16\sqrt{3}$, (ii) $2 + 3i$
 17. (i) -1 , (ii) $1 + 2i, 2\sqrt{5}$
 18. (i) $\sqrt{2}, -\pi/4; \frac{1}{2} + \frac{1}{2}i, 8i$.
 20. (i) $2 - i, 3 - 4i$; (2, -1)
 21. (ii) $3x^2 + 3y^2 + 10x + 3 = 0$.
 22. (i) $x = 1, y = 2$ or $x = -1, y = -2$
 23. (i) a) $13, -23^\circ$, (b) $1, 90^\circ$; (ii) 12 ;
 (iii) $|z + 1 + i| \geq 4$
 24. $2 \sin \theta; \frac{1}{2} \text{cosec} \theta, -\theta; y = -\frac{1}{2}$. 25. $2\sqrt{2} - 2$.

Answers

3. $1 + 2i; 5\sqrt{2}; -2, -1$
 4. (i) $\pm(1 - 3i)$, (ii) $2 - 2i, 1 + i$
 5. sum of squares of a parallelogram = sum of squares of sides
 6. $\pm 1, \pm i, \pm \frac{1}{\sqrt{2}}(1 - i), \pm \frac{1}{\sqrt{2}}(1 + i)$
 7. (i) $-1 + i\sqrt{3}, 2, 2\pi/3; \frac{\sqrt{3}}{2} + \frac{1}{2}i, 1, \pi/6$
 (ii) $1 \pm 2i$; (a) $p^2 = q - 4$, (b) $2p = q + 5$
 8. (a) $-3\sqrt{3} + 5i; 2\sqrt{13}, 2.38 \text{ rad}$, (b) $5, -20$.
 9. (i) $-1 - i, 3\pi/4$, (ii) $2 - i, 2; -10$.