~~SERIES~~ Sequences

A ~~Sery~~ sequence is a set of numbers written down in a definite ~~order~~ pattern ~~with a simple rule by which the numbers are obtained. The set of numbers obtained is called a sequence~~.

Types of sequences

(1) Periodic sequence.

This is the sequence that repeats itself in a given number of terms e.g $u_n = 2 + (-1)^n$

~~Series~~ Sum of terms of a sequence

Arithmetic progression (A.P)

This is a ~~sery~~ sequence of ~~numbers~~ non zero numbers whose adjacent (consecutive) members differ by a common difference d. ie $a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = $ Constant. The constant is called the common difference denoted as d.

n^{th} term of an A.P

If the first term of the A.P is a and the common difference is d then,

2^{nd} term $= a + d \equiv a + (2-1)d$

3^{rd} term $= (a+d) + d = a + 2d \equiv a + (3-1)d$

4^{th} term $= (a+2d) + d = a + 3d \equiv a + (4-1)d$

5^{th} term $= (a+3d) + d = a + 4d \equiv a + (5-1)d$

$\therefore$ The n^{th} term, $a_n = a + (n-1)d$

Sum of the first n terms of an A.P

If the first term is a, the last term l and the common difference d then;

$$a + a+d + \cdots + l-d + l = \text{sum}$$
$$+\; l + l-d + \cdots + a+d + a = \text{sum}$$
$$n(a+l) = 2(\text{sum})$$
$$\Rightarrow \text{sum} = \frac{n}{2}(a+l)$$

but $l = a + (n-1)d$

$\therefore$ Sum of the first n terms, $S_n = \frac{1}{2}n(a + a + (n-1)d)$

$$\therefore S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$$

Examples

① Find the 30^{th} term of the AP below

$7 + 9 + 11 + \cdots$

$[t_{30} = 65]$

eg for $U_n = (\frac{1}{2})^n$, limit is zero and for $U_n = 3 + \frac{1}{n(n+1)}$ the limit is 3.

③ Divergent sequence: This is the sequence whose values tend to infinity as the number of terms increases

④ Oscillating sequence: This is the sequence whose values move to and fro a given value as the number of terms increases eg $U_n = (-1)^n$ oscillates about -1 and 1

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(2) Find the number of terms in the following A.Ps

(i) $5 + 6 + 7 + \cdots + 15$ (ii) $5 + 8 + 11 + \cdots + 302$

$[n = 11]$ $[n = 100]$

(3) The first four terms of an AP are $5 + 11 + 17 + 23 + \cdots$. Find the

(i) 30^{th} term

(ii) Sum of the first 30 terms.

$[t_{30} = 179$ $S_{30} = 2760]$

Geometric progression (G.P)

This is a ~~sery~~ sequence of ~~numbers~~ non zero numbers in which the ratio of ~~the next~~ a given term to ~~the previous~~ its preceding term is ~~always~~ a constant.

$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}}$ = constant. The constant is called the common ratio denoted as r.

n^{th} term of a G.P

If a, a_n and r are the first term, n^{th} term and the common ratio of the G.P then,

$$\frac{t_2}{a} = r \quad ; \quad t_2 = ar \equiv ar^{2-1}$$

$$\frac{t_3}{t_2} = \frac{t^3}{ar} = r \quad ; \quad t_3 = ar^2 \equiv ar^{3-1}$$

$\therefore$ The n^{th} term, $t_n = ar^{n-1}$

Sum of the first n terms of a G.P

Assuming $r < 1$,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

$$- \quad S_n r = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

$$S_n - S_n r = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$\therefore S_n = \frac{a(1-r^n)}{1-r}$ where S_n is sum of the first n terms.

If however $r > 1$ then;

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

Examples.

① In a G.P the first term is 1 and the common ratio is $\frac{1}{4}$. Find the first four terms of the G.P.

$\left[1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\right]$

② The 3rd term of the G.P is 75 and the 4th term is 375. Find the first term and the common ratio of the G.P.

$[r = 5 \text{ and } a = 3]$

③ In a G.P, the sum of the second and third terms is 12. The sum of the third and forth term is -36. Find the first term and the common ratio of the G.P.

$[r = -3 \text{ and } a = 2)$

④ The 2nd, 3rd and 9th terms of an A.P form a G.P. Find the common ratio of the G.P.

$[r = 6]$

In a G.P, the first term is 1 and the sum of a_3 and a_5 = 90. Find r.

$[r = \ ]$

⑤ Given that 31, x and y are consecutive terms of an AP while y, 4 and x are consecutive terms of a G.P, calculate the possible values of x and y.

$[x = 16 \text{ and } y = 1 \text{ or } x = -0.5 \text{ and } y = -32]$

Infinite G.P

Consider the G.P $1 + \frac{1}{2} + \frac{1}{4} + \cdots$. The sum of n terms is given by:

$$S_n = \frac{a(1-r^n)}{1-r}$$

but, $a = 1$ and $r = \frac{1}{2}$

$$\Rightarrow S_n = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$$

$$= 2\left[1-(\tfrac{1}{2})^n\right]$$

as $n \to \infty$, $(\frac{1}{2})^n \to 0$

$$\therefore S_\infty = \frac{a}{1-r}$$

As shown, as $n \to \infty$, $(\frac{1}{2})^n \to 0$. $(\frac{1}{2})^n$ can be taken as zero if n approaches as closely

6 and q are consecutive terms of a G.P.
(i) Express q in terms of p.
(ii) Show that $p^2 - 20p + 36 = 0$ hence determine the values of p and q.

nth term as integral power of 2.

Soln

$S_n = \frac{a(r^n-1)}{r-1}$

$a = 4$

$r - 1 = 3$, $r = 4$

C.f $S_n = \frac{4}{3}(4^n - 1)$

$\therefore$ nth term $= ar^{n-1} = 4 \times 4^{n-1} = 2^{2n}$

OR

For $n=1$, $a = \frac{4}{3}(4-1)$

For $n=2$, $ar = \frac{4}{3}(16-1)$...

$r = 4$, $a = 4$

as we please to a certain value as n increases.

$\Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$

In this case 2 is taken as the sum to infinity.

In general; $a + ar + ar^2 + \cdots = a/_{1-r}$

$$\therefore S_\infty = \frac{a}{1-r}, \text{ when } r > 1$$

$$\text{or } S_\infty = \frac{a}{r-1}, \text{ when } r < 1$$

Examples

Express the following as fractions in their lowest terms.

(a) $0.0\dot{7}$ (b) $0.\dot{4}\dot{5}$

Soln

(a) $0.0\dot{7} = 0.07777 \cdots$

$$= \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots$$

$$a = 7/_{100} \quad r = 1/_{10}$$

$$\therefore 0.0\dot{7} = \frac{7/100}{(1-\frac{1}{10})}$$

$$= \frac{7}{100} \times \frac{10}{9}$$

$$= 7/_{90}$$

Arithmetic and geometric mean

If a, b and c are consecutive terms of an A.P, then b is the arithmetic mean of the A.p

by definition; $c - b = b - a$

$$2b = \frac{c+a}{2}$$

$$\therefore b = \frac{a+c}{2}$$

If a, b and c are consecutive terms of a G.P, then b is the geometric mean of the G.p.

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by definition; $\frac{b}{a} = \frac{c}{b}$

$$\Rightarrow b^2 = ac$$

$$\therefore b = \pm\sqrt{ac}$$

Examples

① The arithmetic mean of a and b is 4. Given that $a = 6$, find the;
(a) arithmetic mean of a^2 and b^2.
(b) geometric mean of a and b

Exercise

① In an A.P the 3^{rd} term is 4 and the 8^{th} term is 49. Find the sum of the first 10 terms

② In an AP the 13^{th} term is 27 and the 7^{th} term is three times the second term. Find the first term and the common difference.

③ If $8-x$, $3x$ and $4x+3$ are adjacent terms of an A.P, find x.

④ The sum of the first n terms of a certain sery is n^2+5n for all integral values of n. Find the first 3 terms and show that the sery is an A.P when $n=1$ and $S=6$.

⑤ If $1/(b+c)$, $1/(c+a)$ and $1/(a+b)$ are consecutive terms of an A.P, prove that a^2, b^2 and c^2 are also in an A.P

⑥ The 3^{rd}, 5^{th} and 8^{th} terms of an A.P are $3x+8$, $x+24$ and x^3+15. Find the value of x and hence deduce the common difference of the A.P

[$x = 3$ or $x = -1.5$]

and the 6^{th} term is 16. Find the sum of the first terms of the A.P.

(19) The sum of the first n terms of a G.P is $\frac{4}{3}(4^n$ ... Find the n^{th} term as an integral power of 2.

... SHS 1.6M. He deposited it to open a deposit account at a local bank. If the bank operates deposit accounts at 9% interest per year, calculate the amount of money he would have after four years given that he does not withdraw any money during this period.

... $u_9 = 577$, show that $2u_8 + u_7 = 577$ and $u_8 - 2u_7 = 41$ hence find u_9

Insert three geometric means between 162 and 1250.

Soln

$a = 162$ and last term $= 1250$. $\Rightarrow 1250 = 162r^4$, $r = \pm\frac{5}{3}$. Terms are 162, ± 270, 450, ± 750, 1250.

The sum of 3 numbers in a G.P is 35 and their product is 1000. Find the numbers.

⑦ An A.P has a common difference of 3. If the n^{th} term is 32 and the sum of the first n terms is 185, find the value of n.

⑧ The sum of p terms of an A.P is q and the sum of q terms is p. Find the sum of $p+q$ terms.
$[-(p+q)]$

⑨ The roots of the equation $2x^3 + 3x^2 + kx - 6 = 0$ are in an A.P. Find k

⑩ Given that $3k+1$, k and -3 are consecutive terms of an A.P, find k

11(a) Find the general term U_n for an A.P where $U_3 = 8$ and $U_4 = -17$

(b) Prove that a sequence defined by $U_n = 3n - 2$ is an A.P hence find U_1 and the common difference d.

⑪ The sides of a triangle are in an A.P and its area is three fifths that of an equilateral triangle of the same perimeter. Prove that the sides are in a ratio $3:5:7$.

⑫ Determine the least possible value of m for which the sum of the first $2m$ terms of an A.P exceeds 883.7.

⑬ Find the number x which when added to each of the numbers 21, 27 and 29 produces three numbers whose squares are in an A.P

⑭ In a G.P, the 4^{th} term is 6 and the 8^{th} term is 96. Find the possible values of r and their corresponding values of the first term.

⑮ The first two terms of an A.P and a G.P are alike. The first term of the A.P is 20 and the sum of its first five terms is 80. Find;

(i) The common ratio of the G.P and the common difference of the A.P.

(ii) The difference in the fifth terms of each progression.

⑯ The n^{th} term of a sery is $u_n = a3^n + bn + c$. Given ...

A sequence is given by $u_n = 2u_{n-1} + u_{n-2}$. If $u_6 = 41$ and $u_9 = 577$, show that $2u_8 + u_7 = 577$ and $u_8 - 2u_7 = 41$ hence find u_8 ($u_8 = 239$)

(2) Insert three geometric means ...

...nd the first four terms of the sequence. (3, 7, 15, 31)

If $u_1 = 1$ and $u_2 = 1$, find the 3^{rd}, 4^{th} and 10^{th} term of the sequence. (2, 3, 60)

PROOF BY INDUCTION

The method of proof by induction is applied when dealing with sums of finite series. The method can only be used to prove that a given expression is the required sum but it does not produce the expression itself.

- Basis step $n=1$
- Assumption step assume $n=k$
- Induction stage $n=k+1$

Examples

① Prove by induction that; $1+2+3+\dots+n = \frac{1}{2}n(n+1)$

Soln

When $n=1$,

$LHS = 1$

$RHS = \frac{1}{2}(1)(1+1) = 1$

LHS = RHS hence it holds

when $n=2$,

$LHS = 1+2 = 3$

$RHS = \frac{1}{2}(2)(2+1) = 3$

LHS = RHS hence it holds.

Since it holds for $n=1$ and $n=2$, then it holds for $n=k$.

$\Rightarrow$ when $n=k$,

$1+2+\dots+k = \frac{1}{2}k(k+1)$.

when $n=k+1$

$1+2+\dots+k+k+1 = \frac{1}{2}k(k+1)+k+1$

$= (k+1)\frac{(k+2)}{2}$

$= \frac{1}{2}(k+1)(k+2)$

$= \frac{1}{2}(k+1)[(k+1)+1]$

but $k+1=n$ and $k+2=n+1$

$= \frac{1}{2}n(n+1)$

$\therefore$ In general (since it applies for $n=1,2,\dots k, k+1$) it applies for all positive integral values of n.

$1+2+3+\dots+n = \frac{1}{2}n(n+1)$

② Prove by induction that, $1^3+2^3+3^3+\dots+n^3 = \frac{1}{4}n^2(n+1)^2$

Soln

when $n=1$,

$LHS = 1^3 = 1$

$RHS = \frac{1}{4}(1)^2(2)^2 = 1$

LHS = RHS hence it holds.

when $n=2$,

$LHS = 1^3+2^3 = 9$

$RHS = \frac{1}{4}(2)^2(3)^2 = 9$

LHS = RHS hence it holds.

Since it holds for $n=2$ and $n=1$, then it holds for $n=k$

$\Rightarrow$ when $n=k$,

$1^3+2^3+\dots+k^3 = \frac{1}{4}k^2(k+1)^2$

when $n=k+1$

$1^3+2^3+\dots+k^3+(k+1)^3 = \frac{1}{4}k^2(k+1)^2+(k+1)^3$

$= (k+1)^2(\frac{1}{4}k^2+k+1)$

$= \frac{1}{4}(k+1)^2(k+2)^2$

but $k+1=n$ and $k+2=n+1$

$\therefore$ In general;

$1^3+2^3+3^3+\dots+k^3 = \frac{1}{4}n^2(n+1)^2$

③ Prove by induction that,

$$\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$$

Soln

$2^0+2^1+2^2+2^3+\dots+2^{n-1} = 2^n-1$

when $n=1$

$LHS = 2^0 = 1$

$RHS = 2^1-1 = 1$

LHS = RHS hence it holds.

when $n=2$

(i) Prove by induction that; $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ where n is whole number.

(ii) ... $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$. Obtain an expression for $\sum_{r=1}^{n}(r+1)(2r+1)$. Hence find the sum of the series $2\times3+3\times5+4\times7+\dots+21\times41$.

Summing series

Prove by induction that $a+ar+ar^2+\dots+ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)$

Prove by induction that

(i) $\frac{3}{4}+\frac{5}{36}+\dots+\frac{2n-1}{n^2(n-1)^2} = 1-\frac{1}{n^2}$ for $n \geq 2$

(ii) $1\times2+2\times5+3\times10+\dots+n(n^2+1) = \frac{1}{4}n(n+1)(n^2+n+2)$

Prove by induction that

* Prove that $3+7+11+\dots+(4n-1) = n(2n+1)$ for all +ve ...
* Prove by induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n} = \frac{2^n-1}{2^n}$

Example

$LHS = 2^0 + 2^1 = 3$

$RHS = 2^2 - 1 = 3$

LHS = RHS hence it holds.

Since it holds for $n=1$ and $n=2$, then it holds for $n=k$.

$\Rightarrow$ when $n=k$,

$2^0+2^1+2^2+2^3+\dots+2^{k-1} = 2^k-1$

when $n=k+1$,

$$2^0+2^1+2^2+\dots+2^{k-1}+2^k = (2^k-1)+2^k$$
$$= 2\cdot 2^k - 1$$
$$= 2^{k+1}-1$$

but $k+1 = n$

hence in general;

$$\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$$

Alternatively;

when $n=1$,

$LHS = 2^0 = 1$

$RHS = 2^1 - 1 = 1$

LHS=RHS hence it holds

when $n=2$

$LHS = \sum_{r=1}^{2} 2^{r-1} = 2^0 + 2^1 = 3$

$RHS = 2^2 - 1 = 3$

LHS=RHS hence it holds.

Since it holds for $n=1$ and $n=2$ then, it holds for $n=k$

$\Rightarrow$ when $n=k$,

$$\sum_{r=1}^{k} 2^{r-1} = 2^k - 1$$

when $n=k+1$,

$$\sum_{r=1}^{k+1} 2^{r-1} = 2^k - 1 + 2^k$$
$$= 2\cdot 2^k - 1$$
$$= 2^{k+1} - 1$$

but $n = k+1$

$\therefore$ In general; $\sum^{n} 2^{r-1} = 2^n - 1$

Prove by induction that; $1\cdot 3 + 2\cdot 4 + \dots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$ for all integral values of n.

④ Prove by induction that $6^n - 1$ is divisible by 5 for all positive integral values of n.

Soln.

Let $n=1$

$$6^n - 1 = 6^1 - 1$$
$$= 6 - 1$$
$$= 5$$
$$= 5(1)$$

let $n=2$

$$6^n - 1 = 6^2 - 1$$
$$= 36 - 1$$
$$= 35$$
$$= 5(7)$$

Since it is true for $n=1$ and $n=2$ then it holds for $n=k$.

$\Rightarrow$ when $n=k$,

$$6^n - 1 = 6^k - 1$$
$$= 5(A)$$

$\Rightarrow 6^k = 5(A)+1$

let $n=k+1$

$$6^n - 1 = 6^{k+1} - 1$$
$$= 6(6^k) - 1$$
$$= 6[5A+1] - 1$$
$$= 30A + 5$$
$$= 5(6A+1)$$

let $6A+1 = L$, ... $5L$ is divisible by 5.

Since $5(6A+1)$ is a multiple of 5 then $6^n - 1$ is always divisible by 5.

Exercise

① prove by induction that $8^n - 7n + 6$ is always divisible by 7.

② Prove by induction that $7^n + 2^{2n+1}$ is ...

③ Prove by induction that $2^n + 3^{2n-3}$ is always ...

Prove that $6^n + 4$ is always divisible by 5 for all ...

$7^n - 2^n$ is divisible by 5 for all n.

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