TRANSFORMATIONS

A transformation is an operation that maps a set of points onto a second set of points. If the first set of points defines a geometric figure (object), then, even the second set will also produce a geometric figure (image). Some transformations preserve the shape and size of the original figure while others distort it.

The main types of Transformations include; Translation, Reflection, Rotation and Enlargement

Any point, line, angle or other feature which do not change in a transformation are called **invariant**

A transformation which does not change size and shape of an object i.e. one in which the object and image are **Congruent** is called **Isometry.** Translations, reflections and rotations are Isometries

TRANSLATIONS

A transformation which maps every point p(x,y) onto a new point p'(x',y') under vector relation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$, where **a** and **b** are fixed numbers, is called a translation.

$\begin{pmatrix} x \\ y \end{pmatrix}and\begin{pmatrix} x' \\ y' \end{pmatrix}$ are the position vectors of P and P' respectively. The position vector of a point is its displacement from the origin.

The translation vector,$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix}$.

Example 1:

The vertices A(3 ,1), B(-2 ,5) and C(-4 ,-3) of triangle ABC are mapped onto A’, B’ and C’ of triangle A’B’C’ respectively by a translation T $\binom{3}{2}$. Determine the coordinates of A’ ,B’ and C’

<u>Approach:1</u>

Solution:

Let the coordinates of A’ be (x’,y’)

$$\binom{x'}{y'} = \binom{3}{1} + \binom{3}{2} = \binom{6}{3},$$

A’ (6, 3)

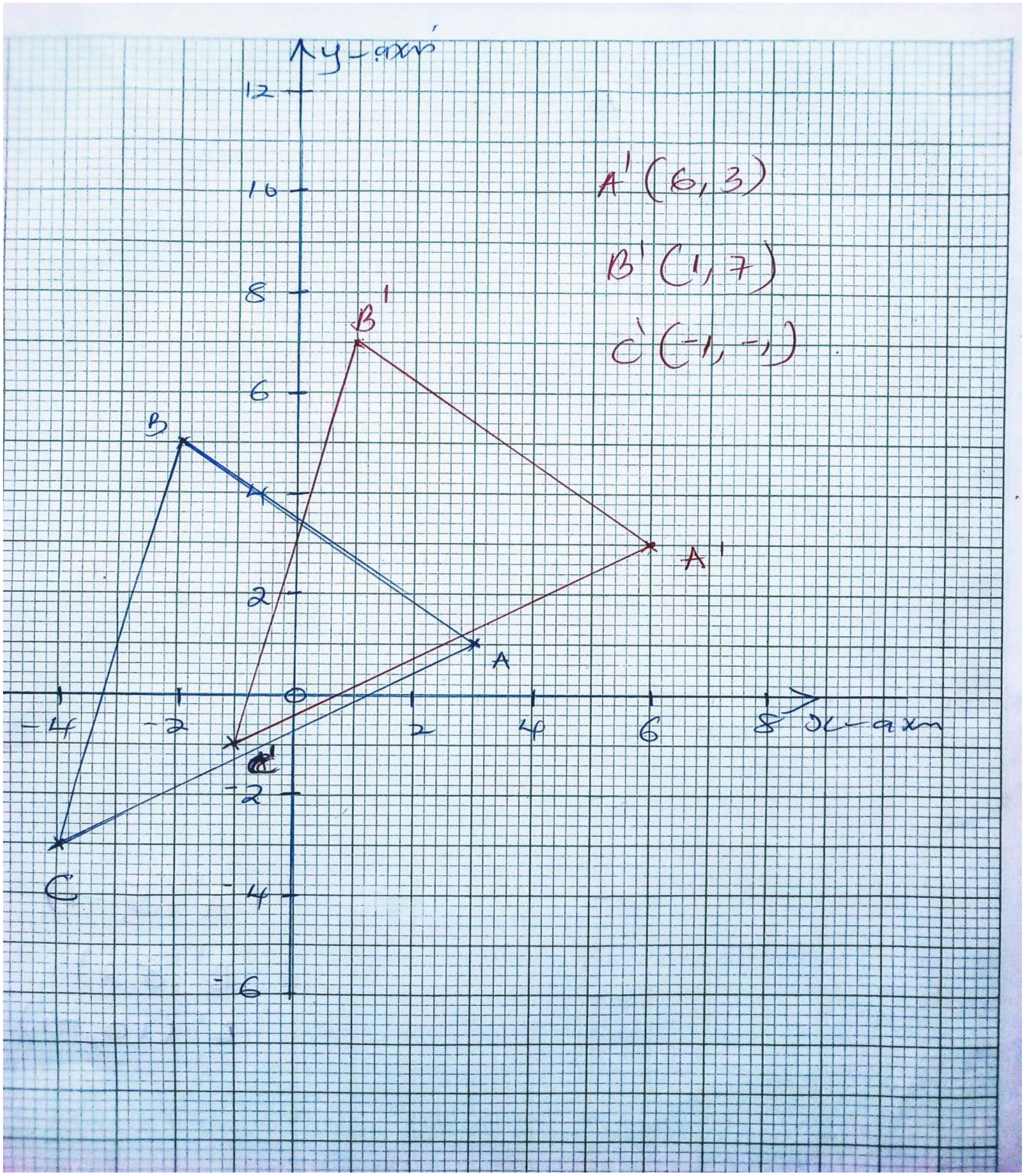
For B’ (x ,y’), $\binom{x'}{y'} = \binom{-2}{5} + \binom{3}{2} = \binom{1}{7},$

B’ (1, 7)

For C’ (x ,y’),$\binom{x'}{y'} = \binom{-4}{-3} + \binom{3}{2} = \binom{-1}{-1},$

C’ (-1 ,-1)

Geometric method:



Example 2:

The vertices A(3 ,1), B(-2 ,5), and C(-4 ,-3) of a triangle ABC are mapped onto A', B' and C' of a triangle $A'B'C'$ respectively by a translation T = $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. **Calculate** the coordinates of A', B' and C'.

Solution

The translation relation is $\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$

The position vector of A' is $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$, A'(6,-1)

The position vector of B' is $\begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, B'(1,3)

The position vector of C' is $\begin{pmatrix} -4 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, C'(-1,-5)

Therefore the coordinates of A', B' and C' are (6 ,1), (1 ,3) and (-1 ,-5) respectively.

Note that these answers may be obtained geometrically.

Example 3: Find the coordinates of A' , B' and C', the image of vertices of the object ABC under translation given by the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, if the coordinates of A, B and C are (1, 2), (4, 5), (6, 2) respectively.

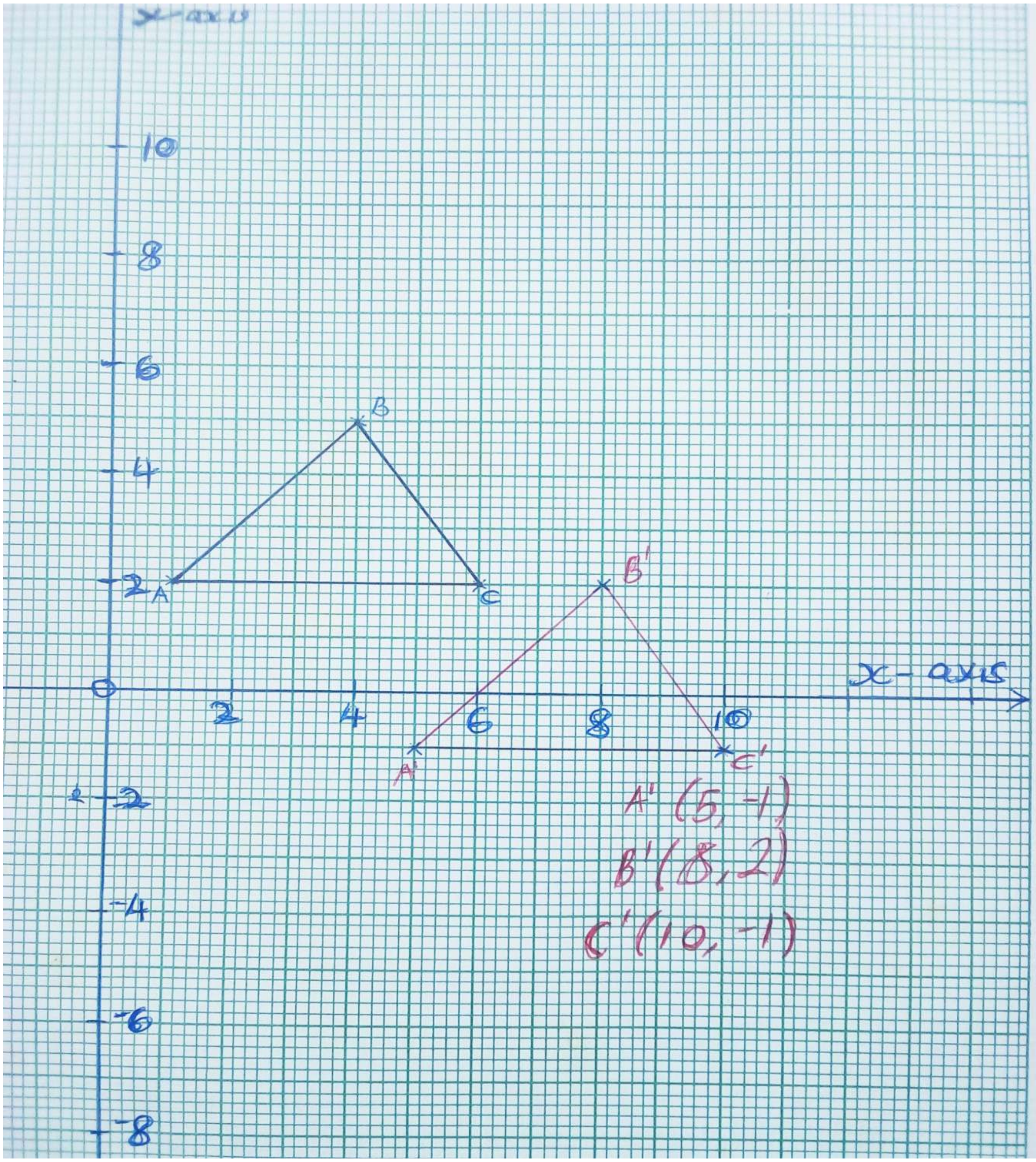
Approach:1

For A', $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, A' (5, -1)

B', $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, B' (8, 2)

C', $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$,C' (10, -1)

Geometrical:



Successive translations

Two or more translations may be subjected to an object, one after another.

Example 1:

A triangle PQR with coordinates P(4,0),Q(2,3) and R(-2,1) is translated by $T_1 = \binom{2}{5}$ to give P'Q'R' and P'Q'R' is then translated by $T_2 = \binom{1}{-3}$ to give P"Q"R". Determine the

i) coordinates of the vertices of both image triangles
ii) single translation T that maps PQR to P"Q"R"

Solution

i) For P' $\binom{x'}{y'} = \binom{4}{0} + \binom{2}{5} = \binom{6}{5}$ P'(6,5)

Q' $\binom{x'}{y'} = \binom{2}{3} + \binom{2}{5} = \binom{4}{8}$ Q'(4,8)

R' $\binom{x'}{y'} = \binom{-2}{1} + \binom{2}{5} = \binom{0}{6}$ R'(0,6)

For P" $\binom{x''}{y''} = \binom{6}{5} + \binom{1}{-3} = \binom{7}{2}$ P"(7,2)

Q" $\binom{x''}{y''} = \binom{4}{8} + \binom{1}{-3} = \binom{5}{5}$ Q"(5,5)

R" $\binom{x''}{y''} = \binom{0}{6} + \binom{1}{-3} = \binom{1}{3}$ R"(1,3)

ii) The single translation $T = T_1 + T_2$

$T = \binom{2}{5} + \binom{1}{-3}$

$T = \binom{3}{2}$.

Example 2:

All the points P(x ,y) in the x-y plane are subjected successively to a translations given by vectors $T_1 = \binom{-2}{1}$, $T_2 = \binom{3}{1}$, $T_3 = \binom{4}{5}$

Find the single translation

a) T which will map all points P(x,y)to their final positions

b) T' which will restore all the points P(x,y) back to their original positions.

Solution

a) $T = T_1 + T_2 + T_3$

$T = \binom{-2}{1} + \binom{3}{1} + \binom{4}{5}$

$T = \binom{5}{7}$

b) $T' = -T$

$T' = \binom{-5}{-7}$

<u>EXERCISE</u>

Qn1. Find the coordinates of A', B' and C', the images of the vertices of triangle ABC under a translation given by $\binom{4}{-3}$, if the coordinates of A, B and C are (1,2), (4,5) and (6,7) respectively.

Qn2. The position vector of P is $\binom{2}{3}$. Under a certain translation, P is mapped onto the point P' which has position vector $\binom{4}{-2}$. Under the same translation, Q is mapped onto point Q' which is (3, 6). Find the coordinates of Q.

Qn3. A point P(-2,1) undergoes two translations T_1 and T_2 successively where $T_1=\binom{2}{-1}$ and $T_2=\binom{3}{-1}$ to give two images P'(x',y') and P''(x'',y''). Determine the coordinates of P' and P''. Hence find a single translation that maps P'' back to P.

REFLECTION

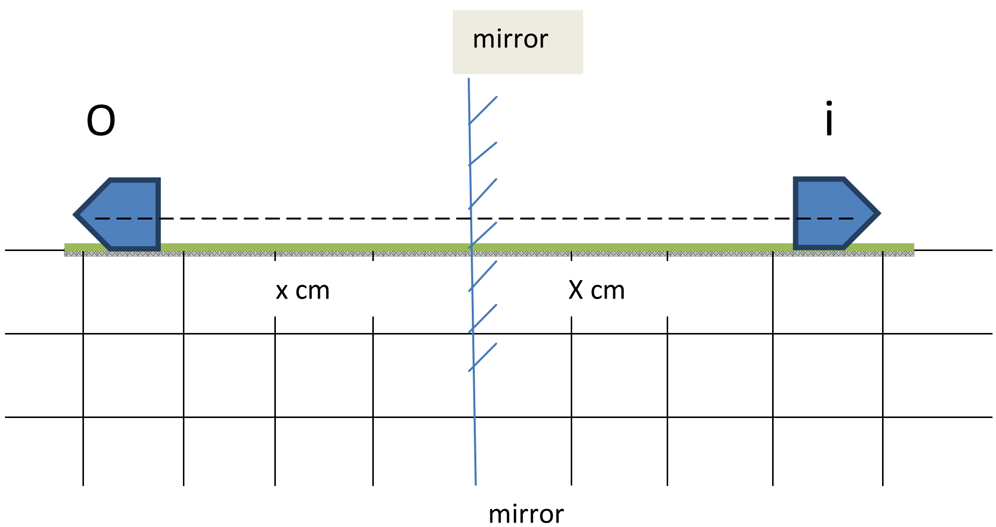
In this type of transformation, the image is congruent to the object. In this case the object and the image will be facing each other

A reflection transformation is an isometry since the shape and size of the object are restored

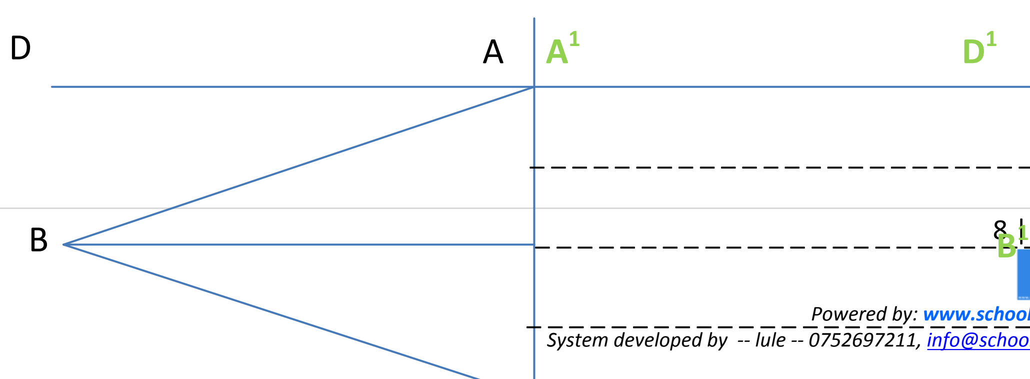
This type of Transformation takes place in presence of a mirror/reflecting surface

Properties of reflection

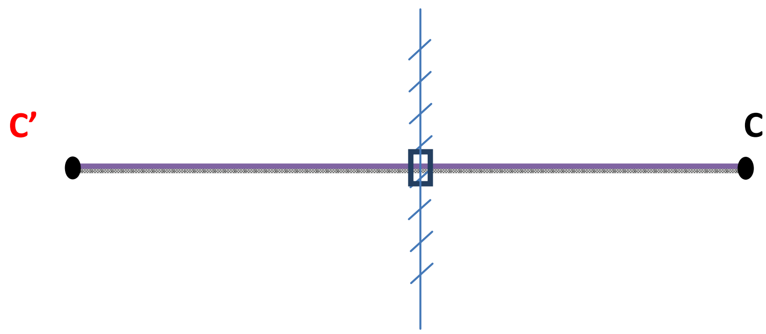
a). The mirror (reflecting surface) is between the object and the image at equal distance from them.



b). The points which are on the mirror line are invariant.



c). The line joining a points to its image is perpendicular to the mirror / reflection line



<u>Reflection by construction if the mirror line is known</u>

Procedure:

*Draw a perpendicular line from the object point to the mirror (reflection line) and extrapolate it to the opposite side of the mirror.

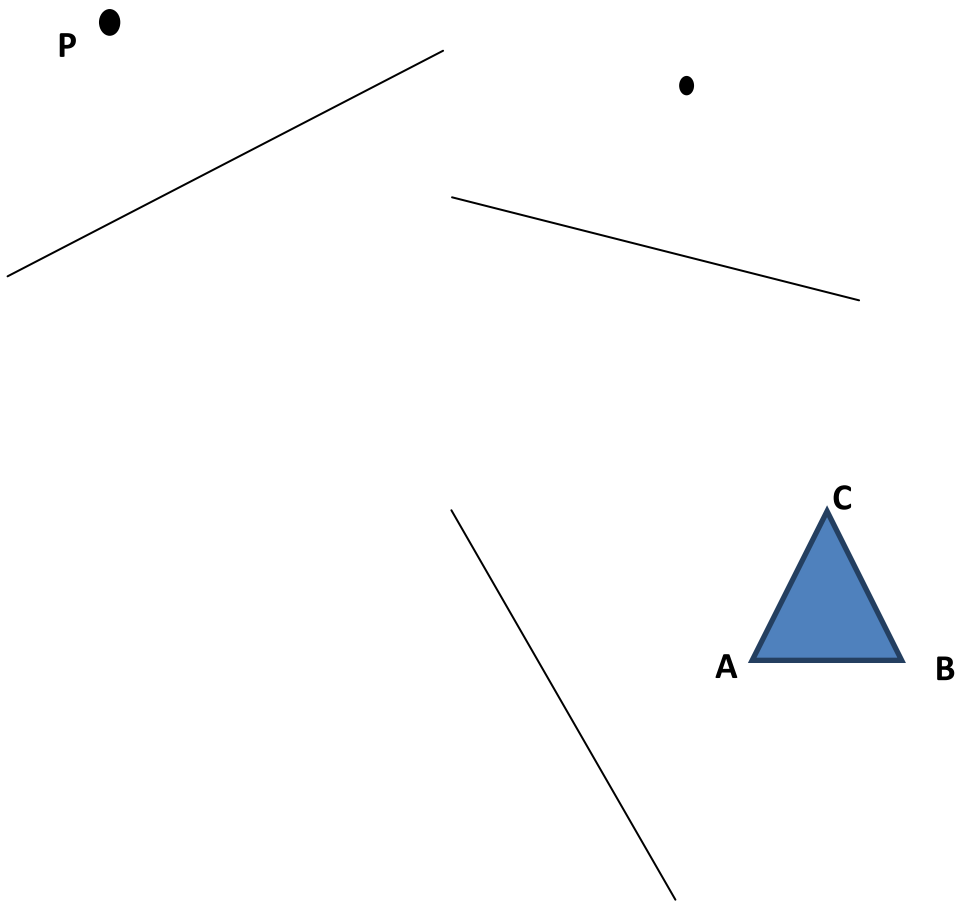
*With your compass, measure, draw the object distance from the mirror line on the opposite side

*Locate the image point.

<u>Example:</u>

Find the images of the following objects under the given reflection line.

A


P
C
A
B

Finding the reflection line

Procedure:

*Given the points and its image to find the position of the reflection line, join the point to its image i.e. A to A' to form AA'.

*Draw a perpendicular bisector of AA'and this perpendicular bisector is the reflection line.

Note:

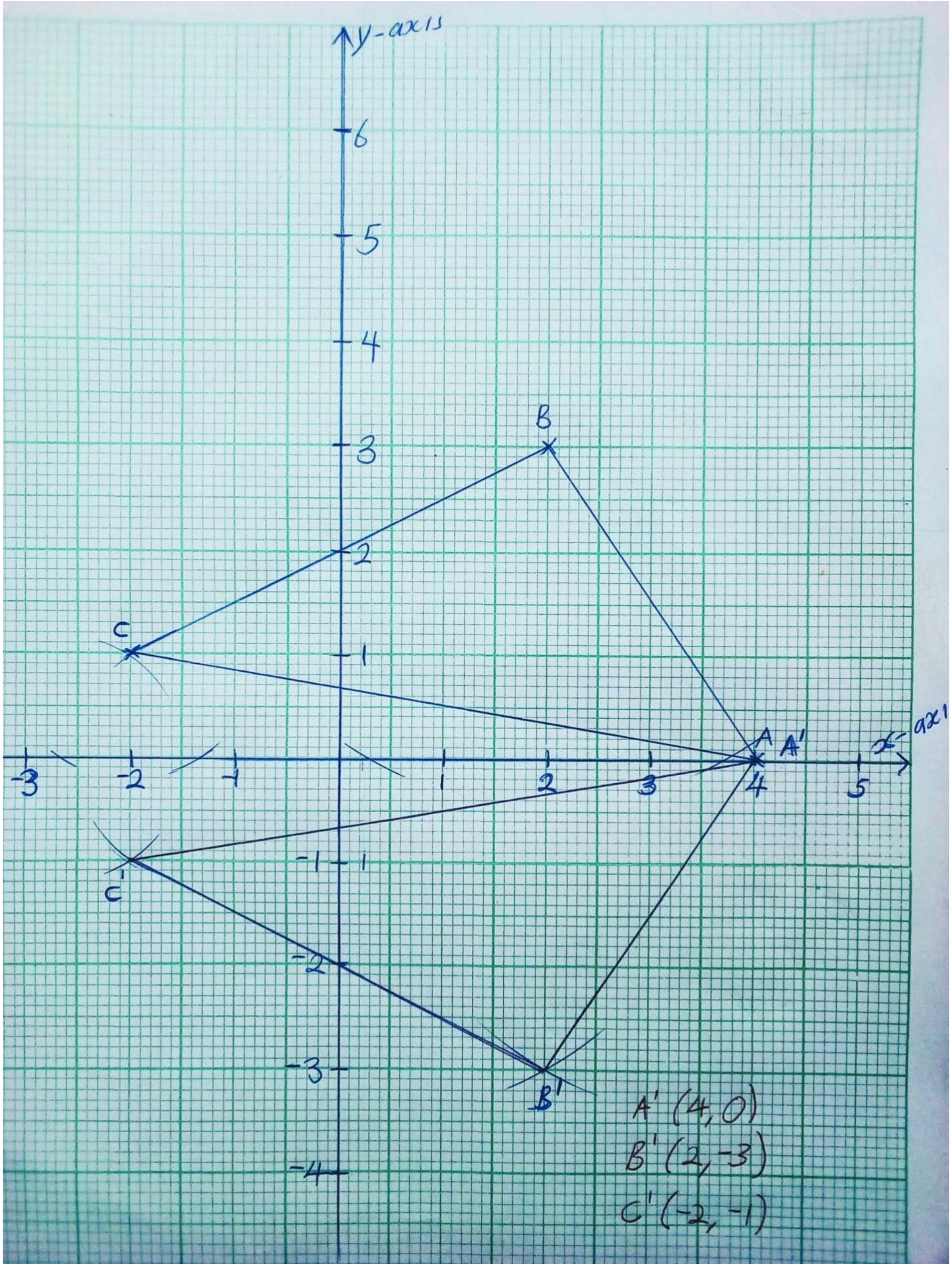
Even with a plane figure like a triangle ABC, choose a point and its image and follow the above procedure

Reflection on a Cartesian plane.

For reflection on a Cartesian plane the mirror is always called a reflection line. The properties of a mirror still stand and have to be used to find images objects or reflection lines.

Example:

A triangle ABC with coordinates A(4, 0), B(2, 3) and C(-2, 1), is reflected in the x-axis. Draw the object and the image and the give the invariant point.



The invariant point is A(4,0)