UCE

Mathematics 2

(For S.2)

BY

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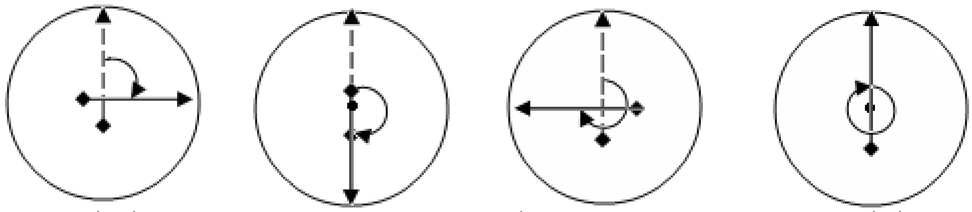
Chapter 1.

ROTATION

There are many objects that turn when force is applied to them. Examples are doors, the cover of a desk and so on. The turning of an object about a fixed point or axis is called **rotation.** The amount of turning is called the **angle of rotation**. The angle of rotation is measured in degrees. A rotation of 360^0 is called a **revolution**. A rotation of 180^0 is called a **half turn** and a rotation of 90^0 is called a **quarter turn**. Figure 7.1 shows the various turns. The dotted lines show the initial positions. State the angle of rotation in each case.

Fig. 7.1



Direction of rotation

The direction in which the hands of a clock turn is called **clockwise** and the rotation in the opposite direction is called **anticlockwise.**
The angle of rotation in an anticlockwise direction is **positive** and that in a clockwise direction is **negative**. Thus, a rotation of 90^0 anticlockwise is written as $+90^0$ and 90^0 clockwise as -90^0.

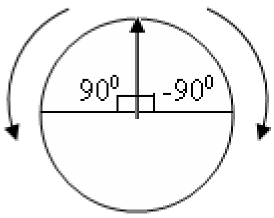


Fig. 7.2

Properties of rotation

When an object is given a rotational transformation, the image is always the same size as the object. Such a transformation is called an **isometry**.
Note: *A rotation is fully described by stating the angle of rotation and centre of rotation*.

Example 1

(a) Find the position of line AB when rotated through 60^0 about point C.

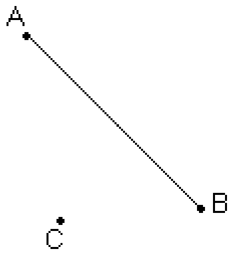


Fig. 7.3

(b) Find the position of triangle PQR when rotated through an angle of -150^0 about point O.

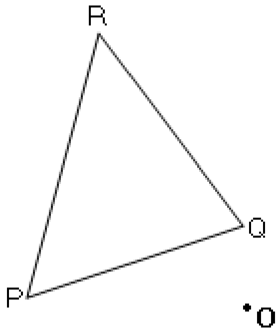


Fig. 7.4

Solutions

(a) Join B to C and measure CB. Use a protractor and a ruler to mark a point B′ such that ∠ BCB′ **=** 60^0 and BC = B′C. Similarly, join A to C and measure an angle such that ∠ACA′ = 60^0 and CA′ = CA. Join A′ to B′ to obtain the image of line AB.

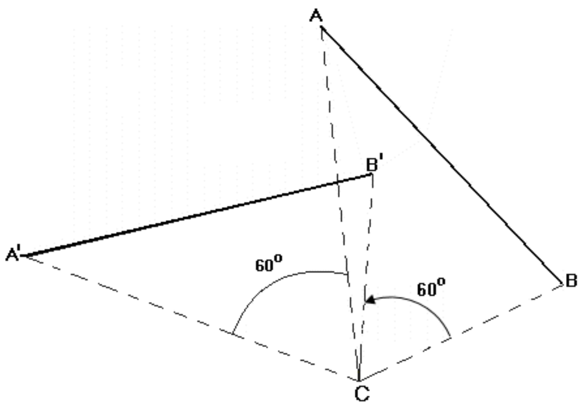


Fig. 7.5

Note: *When the perpendicular bisectors of AA^1 and BB^1 are drawn, they meet at the centre of rotation.*

(b) Join P to O and measure PO. Use a protractor and a ruler t mark point P^1 such that $\angle POP^1 = -150^0$ and $OP^1 = PO$. Join Q to O, mark point Q^1 such that $\angle QOQ^1 = -150^0$ and $QO = OQ^1$. Join R to O, and mark point R^1 such that $\angle ROR^1 = -150^0$ and $RO = OR^1$. Finally join points P^1, Q^1, and R^1 to obtain the image of triangle PQR.

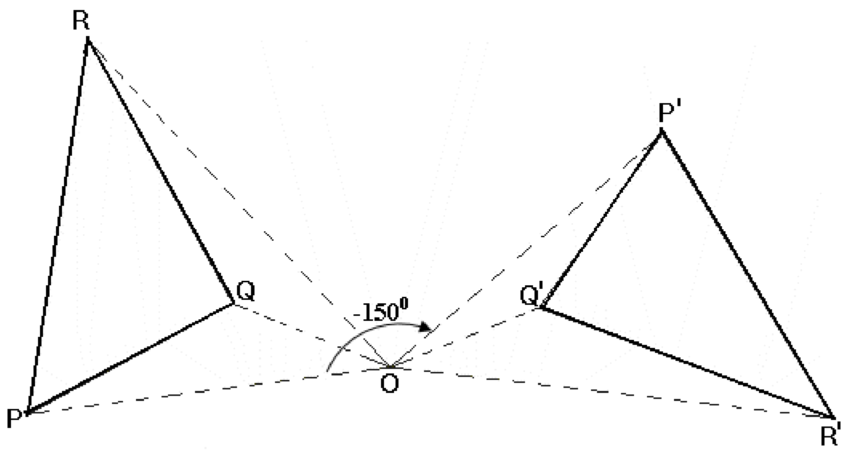


Fig.7.6

Centre and angle of rotation

Given an object and its image, the centre of rotation is found by joining at least two points on the object to the corresponding points on the image. Construct perpendicular bisectors to each line such that the bisectors meet at a point. The point where they meet is the centre of rotation.
To determine the angle of rotation, join a point on the object and its corresponding point on the image to the centre of rotation. Measure the angle between the two lines. This is the angle of rotation.

Example

Triangle ABC is mapped onto triangle $A^1B^1C^1$ under a rotation. Find the:
(a) centre of rotation (b) angle of rotation.

Solution:

(a) Draw a perpendicular bisector to lineAA^1. Similarly, draw a perpendicular bisector to line BB^1. Let the two bisectors meet at a point O.
O is the centre of rotation.

(b) Join O to A and O to A^1. Measure angle AOA^1. $AOA^1 = 90^0$ or -270^0. This is the angle of rotation.

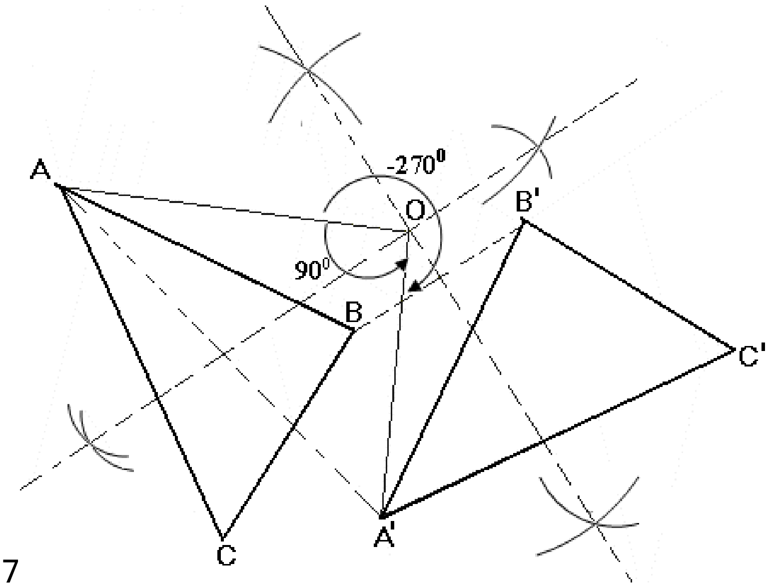
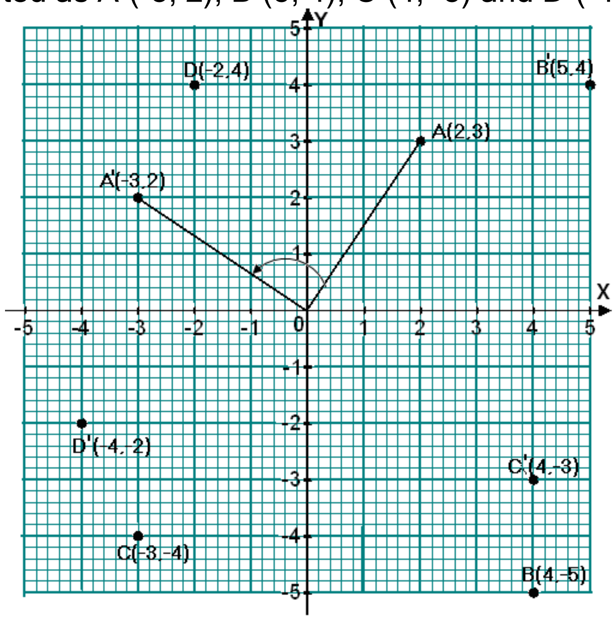


Fig. 7.7

Rotation in the Cartesian plane

The figure below shows points A(2, 3), B(4, -5), C(-3, -4) and D(-2, 4) plotted on squared paper. Each object point is then rotated about the origin O(0, 0) and its image point noted as $A^1(-3, 2)$, $B^1(5, 4)$, $C^1(4, -3)$ and $D^1(-4, -2)$.



Rotation of $+90^0$ about point (0, 0)

Join each point in the figure above to the origin O(0, 0) and rotate this line through +90 about point O(0, 0). You will notice that:

- A(2, 3) is mapped onto (-3, 2),
- B(-4, 5) is mapped onto (5, 4),
- C(-3, -4) is mapped onto (4, -3),
- D(-2, 4) is mapped onto (-4, -2).

It follows that a point P(m, n) will be mapped onto $P^1(-n, m)$ under a rotation of $+90^0$ about (0, 0).

Rotation of -90^0 about (0, 0)

Using the same points in the figure above, when each point is rotated through -90^0 about (0, 0):

- A(2, 3) is mapped onto (3, -2),
- B(4, -5) is mapped onto (-5, -4),
- C(-3, -4) is mapped onto (-4, 3),
- D(-2, 4) is mapped onto (4, 2).

In general, given a point P(m, n) will be mapped onto P^1(n, -m) under a rotation of -90^0 about point (0, 0).

Rotation of 180^0 about point (0, 0)

If each point in the above figure is rotated through 180^0 about (0, 0), the image of:

- A(2, 3) is (-2, -3),
- B(4, -5) is (-4, 5),
- C(-3, -4) is (3, 4),
- D(-2, 4) is (2, -5).

In general, a point P(m, n) becomes P^1(-m, -n) under a rotation of 180^0 about point (0, 0).

Note: Under a rotation, the size and shape of the object and its image are the same. In this case, the object is said to be **congruent** to the image.

In the figure below, triangle PQR is rotated through (a) $+90^0$ (b) -90^0 and (c) 180^0 about point O(0, 0). The images are shown in the figure below as (a), (b) and (c).

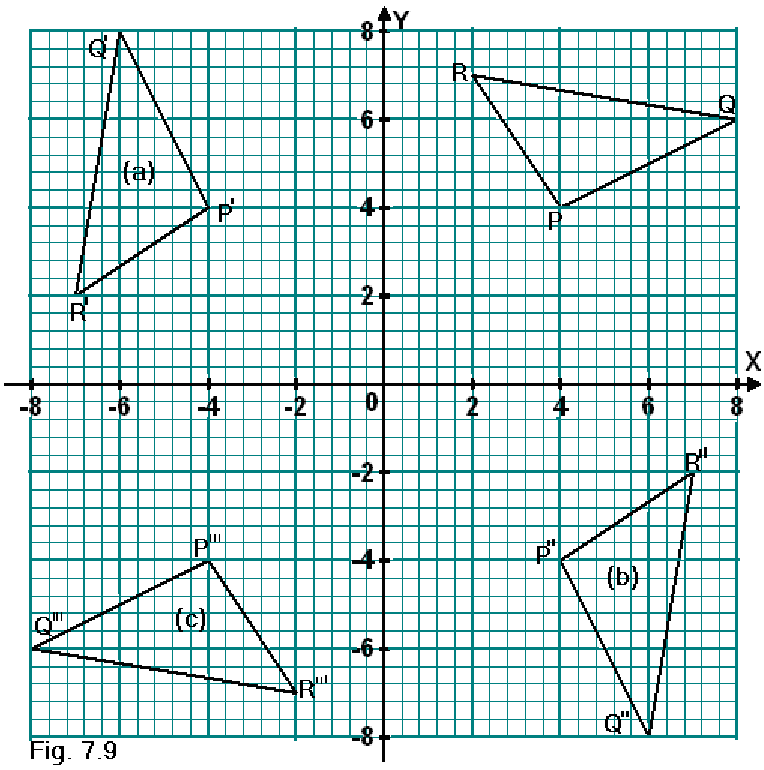


Fig. 7.9

Exercise 1.1

1. State the images of the following points under a rotation of $+90^0$ about point O(0, 0).
   (a) (3, 6) (b) (-4, 5)
   (c) (-5, -7) (d) (2, -1)
2. State the images of each of the following points under a rotation of -90^0 about point (0, 0).

   (c) (2, -5) (d) (-5, 6).
3. State the images of each of the following points under a rotation of 180^0 about point (0, 0).
   (a) (-3, 7) (b) (3, -2)
   (c) (10, 5) (d) (-11, -15)
4. A shape is formed by joining the points with coordinates (6, 4), (6, 6), (5, 5) and (6, 5) in that order. Find the coordinates of the images of these points under a rotation of $+90^0$ about point O(0, 0).
5. Use squared paper to find the coordinates of the images of each of the following:
   (a) A(-1, -3), centre (0, -1) under a rotation of -90^0,
   (b) B(4, 0), centre (1, 0) under rotation of $+90^0$,
   (c) C(2, -3), centre (2, 0) under rotation of 180^0
   (d) D(1, 2), centre (3, 4) under a rotation of $+90^0$,
   (e) E(0, 2), centre (0, 3) under a rotation of -90^0,
   (f) F(-1, 2), centre (2, 3) under a rotation of 180^0.
6. A point P(6, 1) is mapped onto P^1(5, 4) and point Q(1, 6) is mapped onto Q^1(-2, 5) under a rotation. Find the centre and angle of rotation.
7. A triangle whose vertices are X(-2, 2), Y(-4, 2) and Z(-6, 4) is mapped onto triangle$X^1Y^1Z^1$ with vertices X^1(2, -1), Y^1(4, -1) and Z^1(6, -3) under a rotation. Find the centre and angle of rotation.

Rotational symmetry

When a plane figure is mapped onto itself under a rotation of angle θ, where θ is between 0^0 and 360^0, it is said to have a rotational symmetry with the centre of rotation being the same as the centre of symmetry.

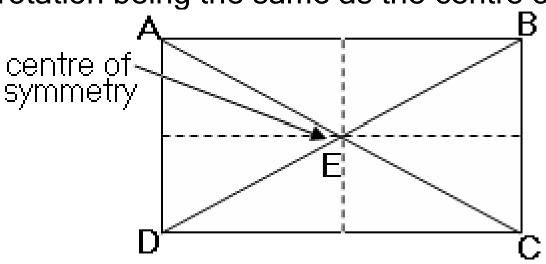


Fig. 7.10

Figure 7.10 represents an outline of a rectangular piece of cardboard with its diagonals meeting at point E.

When the cardboard is placed on the outline and rotated through 180^0 about E, it occupies the same outline. After another rotation of 180^0 about E, the card board assumes the initial position. Thus, the cardboard fits onto the outline in two positions before assuming its initial position. The rectangle is said to have a **rotational symmetry of order 2**. Point E is the **centre of rotation**.

Exercise 1.2

1. Find the images of point P under each of the following situations.

Point P	Angle of rotation	Centre of rotation
(5, 0)	$+90^0$	(1, 0)
(-2, -3)	$+180^0$	(1, 1)
(-3, 4)	-90^0	(0, 2)
(4, 1)	$+180^0$	(0, 0)

2. Under a rotation, point A(7, -2) is mapped onto point A^1(6, 1) and point B(2, 3) is mapped onto point B^1(-2, 5).
Find:
(a) the coordinates of the centre of rotation,
(b) the positive angle of rotation.

3. A triangle with vertices A(-2, 3), B(-5, 3) and C(-7, 7) is mapped onto A^1(2, -1), B^1(5, -1) and C^1(7, 5) respectively. Find the centre of rotation.

4. Point P(2, 5) and Q(9, -6) are mapped onto P^1(8, 1) and Q^1(5, -12) under a rotation. Use squared paper to find:
(a) the coordinates of the centre of rotation,
(b) the angle of rotation,
(c) the coordinates of R if R^1 is (-5, 4).

Chapter 2.

STATISTICS

Statistics is that branch of mathematics which is concerned with the collection, organization, interpretation, presentation and analysis of numerical data. To a statistician, any information collected is called **data**. When data has not been ordered in any specific way after collection, it is called **raw** data.

Representation of data

1. Frequency tables (distributions)

The **frequency** of an event is the number of times that the event occurs. While counting the scores, a tally mark, /, is made for every count and the fifth stroke crosses over the other four to have 𝍸 .

Example .1

Forty people were asked to judge which of five paintings, labeled A, B, C, D and E, was the best. The results are given below.

B	C	B	B	D	A	E	A
C	C	A	A	B	D	E	A
B	D	B	B	B	C	A	D
E	B	C	A	A	C	D	B
C	C	A	D	B	B	D	A

If a prize was given for the picture with the most votes, which picture won?

Solution

First we put the data into a **frequency distribution** (frequency table). This is a table which gives each data item together with the number of times they occur.

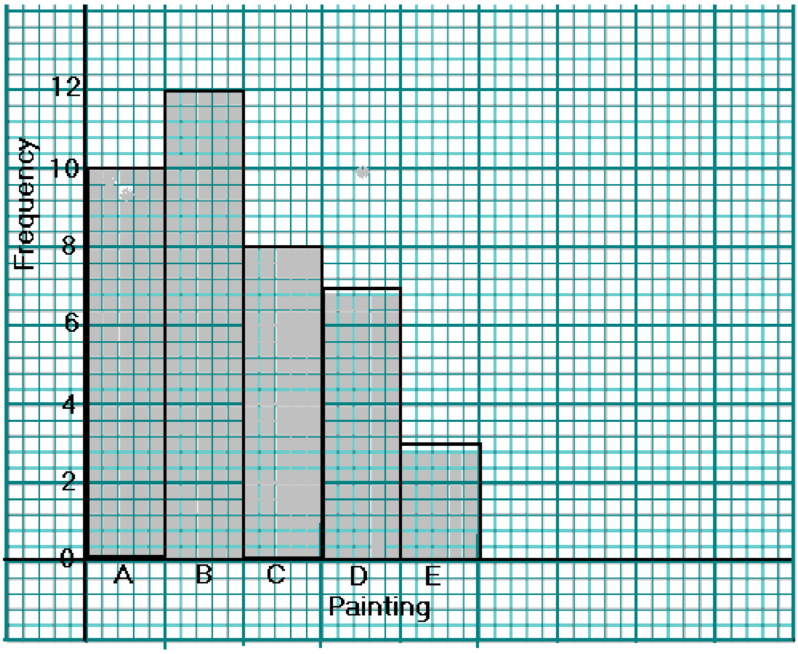
Painting	Tallies	Frequency
A	𝍸 𝍸	10
B	𝍸 𝍸 ||	12
C	𝍸 |||	8
D	𝍸 ||	7
E	|||	3
Total		40

Painting B, with twelve votes, wins the prize.

The above data may be displayed in various ways.

2. Bar chart:

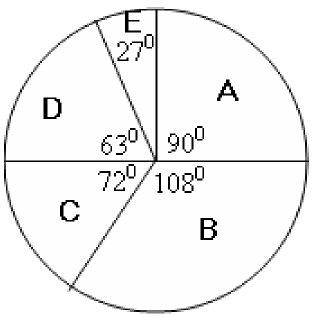
Bar charts or graphs can be used to make quick comparison of data. They are either horizontal or vertical, jointed or disjointed. The height or the length of the bar represents the frequency and its width is not significant.
The bar graph for the above data can thus be drawn as follows.



3. Pie Chart:

In a pie chart, each number is represented by the area of a sector of a circle. The frequency is proportional to the angle of the sector.
The forty votes are represented by 360^0 (a full turn); thus one vote is represented by $360^0 \div 40 = 9^0$.

	Frequency		Angle
A	10	$10 \times 9 =$	90^0
B	12	$12 \times 9 =$	108^0
C	8	$8 \times 9 =$	72^0
D	7	$7 \times 9 =$	63^0
E	3	$3 \times 9 =$	27^0

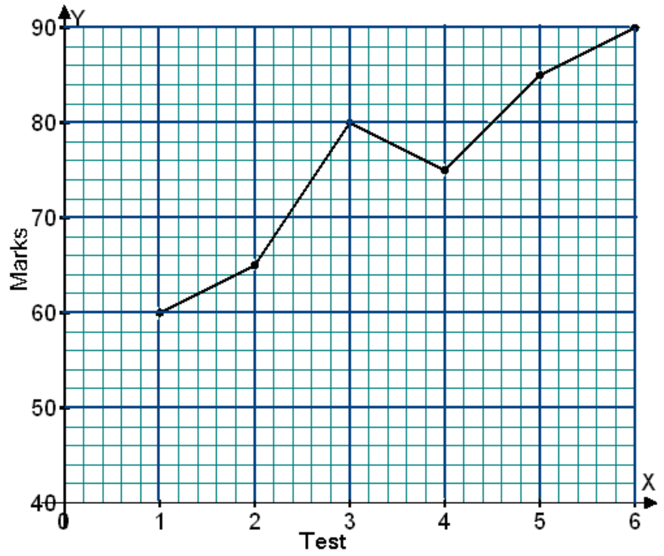


4. Line graphs

A line graph is used to show how things change over a period of time. For example: A student obtained the following marks in six mathematics weekly tests. Draw a line graph to represent this information.

Test	1	2	3	4	5	6
marks	60	65	80	75	85	90

To draw the line graph, mark the test numbers on the horizontal axis and the marks on the vertical axis. Plot the ordered pairs on the graph with dots as shown in the figure below. To obtain the line graph, join the dots with straight lines.



Measures of central tendency

These are values about which the distribution of data in a particular se may be approximately balanced. The common ones include the mean, median and mode.

The mean ($\bar{x}$)

The mean is the value obtained when the total sum of the values of the members in a list or distribution is divided by the total frequency.

Example 1

The masses (in kilograms) of nine students are 42, 47, 52, 57, 62, 67, 72, 77 and 82. Calculate their mean mass.

Solution

Add all the masses together and divide by 9.
The sum of the masses is 558 kg.
The total number of students is 9.

The mean mass is $\frac{558}{9}$ = 62 kg.

Example 2

Find the mean of the following frequency distribution.

x	1	2	3	4	5	6
f	2	4	1	3	3	3

Solution

To calculate the mean of the above data the following steps are followed.

- Multiply each frequency (f) by its corresponding value (x) to obtain the product (f.x)
- Find the sum of the products, Σ(f.x), and then divide this sum by the sum of the frequencies Σ(f) to obtain the mean, i.e. $\bar{x} = \frac{\sum(f.x)}{\sum f}$

x	f	f.x
1	2	2
2	4	8
3	1	3
4	3	12
5	3	15
6	3	18
	Σf=16	**Σf.x = 58**

The mean, $\bar{x} = \frac{58}{16}$ = 3.625.

In statistics, the symbol Σ(sigma) is used to mean the **sum of**. Thus, the sum of the frequencies (f) is written as Σf and the sum of the products (f.x) is written as Σfx.

The median

Median is the middle value of a given set of data when all the entries are arranged in order of size. It is the point at which exactly half of the data are above and half are below.

In ungrouped data, if the total number of members in a list is odd, the median position is a whole number given by $\frac{1}{2}(n+1)^{th}$ member of the group. This formula gives the position at which the median is found in ranked data. To get the median, we go back to the ranked data and look for the value or entry at the position obtained from the calculation. When the position is not a whole number, we get the average of the values just before and just after the one obtained through the computation.

Example 3

Find the median of each of the following lists of numbers.

(a) 1, 7, 1, 6, 2, 5, 2, 5, 2, 5, 3, 5, 3, 4, 4.

(b) 29, 12, 25, 14, 18, 21, 18, 20, 19, 20.

Solutions

(a) Rearranging the numbers in order of size, we have:
1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 5, 5, 6, 7

The number of members (n) is 15. The median is the $\frac{1}{2}(15+1)^{th}$ member. That is the 8th member. From the ordered list, the 8th member is 4. The median is 4.

(b) Rearranging the numbers in order of size, we get:
12, 14, 18, 18, 19, 20, 20, 21, 25, 29
The number of members (n) is 10. The median position is given by
$\frac{1}{2}(10+1) = \frac{1}{2}\times 11 = 5.5$
This means that the median value falls between the 5th and 6th members in the ordered data, and its value is their average.
The 5th and 6th members are 19 and 20 respectively. Therefore, the median is $\frac{19+20}{2} = 19.5$.

The median does not need to be among the numbers in the set of data.

The median of an ungrouped frequency distribution

In order to find the median using cumulative frequencies or the number of members that lie above or below a particular value in a set of data, we must calculate the first value with a cumulative frequency greater than or equal to the median.

Example 4

Find the median of the following distribution.

x	2	3	4	5	6	7	8
f	4	8	20	20	2	2	4

Solution

The total frequency, $\Sigma f = 60$. The median position is $\frac{1}{2}(60+1) = 30.5$. This means the median value lies between the 30th and 31st values in the ordered data and its value is their mean.
In order to get the 30th and 31st values, we need to make a cumulative frequency table as shown below.

x	Freq. (f)	Cumulative frequency (cf)
2	4	4
3	8	12
4	20	32
5	20	52
6	2	54
7	2	56
8	4	60

The cumulative frequency is obtained by adding all the frequencies below and up to and including the particular value. From the table, we can see that the cumulative frequency corresponding to x = 8 is 60. This is the total number of observations in the distribution. From the cumulative frequency it can be seen that the 30th member is 4 and the 31st member is also 4. Therefore, the median is $\frac{4+4}{2} = 4$.

The mode.

The mode of a distribution is the most frequently occurring value. In a frequency distribution, the mode is the value with the highest frequency.

Example 5

Find the mode of the following set of numbers.
2, 4, 4, 5, 7, 7, 7.

Solution

The mode is 7 as this appears three times.

Example 6

Find the mode of the following frequency distribution.

(a)

x	3	4	5	6	7
f	1	3	8	10	4

(b)

x	12	13	14	15	16
f	6	9	9	3	5

Solutions

(a) The mode is 6.
(b) The mode is 13 and 14 as these have the highest frequency.

Exercise 2

1. Find the mode, mean and median of the following sets of numbers
   (a) 3, 5, 6, 8, 9, 9, 9
   (b) 10, 10, 10, 20, 30, 30, 30, 30.
   (c) 7, 1, 3, 5, 2, 9, 1.

2. The mean of 4 numbers is 8. If three of the numbers are 4, 5 and 10, find the fourth number.

3. The total of the heights of twenty-six girls was 41.86 m. Find the mean height of the girls.

4. In a year, a student scored the following marks in tests and examinations.

Term	Test 1	Test 2	Test 3	Exams
1	75	51	62	49
2	54	45	64	74
3	80	60	67	39

Calculate the student's mean score.

5. The number of litres of milk delivered to 50 houses on one morning is shown below.

No. of litres	1	2	3	4	5	6
No. of house	10	14	18	4	2	2

Calculate the mean number of litres delivered per house.

6. The table below shows the marks scored by students in a test.

Mark	0	1	2	3	4	5
No. of students	2	6	15	8	10	2

Calculate the mean and median mark.

7. The lengths, in centimeters, of 10 nails are: 3.01, 2.96, 3.04, 2.98, 3.00, 2.97, 2.99, 3.02, 3.03, and 2.94. Find the median length of the nails.

8. The number of goals scored in super league matches is shown in the table below.

Goals	frequency
0	10
1	20
2	10
3	6
4	9
4	8

Calculate the mean and median number of goals scored. State the modal number of goals scored.

9. A company's costs are split into the following categories.

Wages	45%
Rates	15%
Materials	30%
Transport	10%

Show this information in a pie chart.

10. A group of fifty year olds were asked how much television they had watched on a particular evening. Their answers, to the nearest hour, are shown below.

1	3	0	3	1	4	5	4	3	2
0	4	6	2	2	2	3	2	2	7
3	1	0	2	3	3	0	1	7	2
5	4	2	1	1	3	4	2	4	4
0	0	1	2	1	3	1	2	7	2

(a) Set up a frequency distribution to represent this information.
(b) Draw a bar chart to show the above data.
(c) State the modal time.

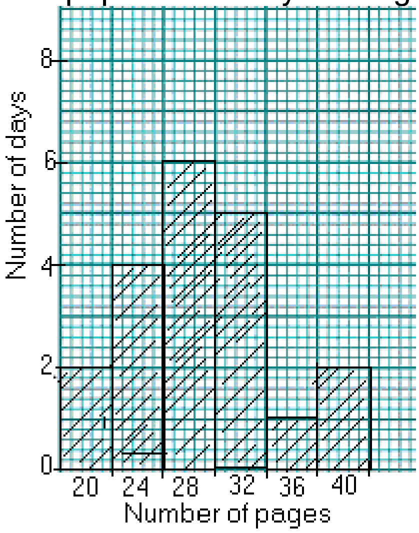
11. Various types of large white loaves are delivered to a supermarket. The number of each type delivered on a particular day is shown below.

Thin sliced 45
Medium sliced 84

Thick sliced 15
Uncut 36

(a) In a pie chart, what angle would represent 1 loaf?
(b) Display the above information in a pie chart.

12. The bar chart below shows the number of pages per day in the New Vision news paper for 20 days during a certain month.



(a) State the modal number of pages.
(b) Calculate the mean and median number of pages.

13. The table below shows the number of letters collected by a school from the post office in the first six weeks of the term.

Week	1	2	3	4	5	6
No. of letters	80	110	120	160	110	120

Show this information in a:
(i) bar graph,
(ii) line graph.

14. The table below shows the production of eggs by 200 hens in 7 days.

Day	1	2	3	4	5	6	7
No. of trays	1	2	2	3	4	5	6

(a) Draw:
(i) a bar graph to show this data,
(ii) a line graph to represent the information.
(b) A tray of eggs costs sh.3, 000. How much money was collected in the 7 days?

15. The table below shows the number of trays of eggs supplied to a hotel by two farmers.

Days	1	2	3	4	5	6	7
Farmer A	8	10	12	15	12	11	8
Farmer B	10	12	11	12	14	16	18

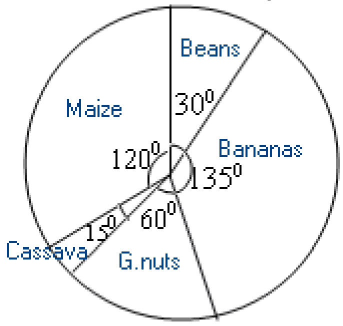
Draw line graphs to represent this data.

16. After a national examination at the end of secondary school education, students from six schools were selected to attend university as follows:

School	A	B	C	D	E	F
No. of students	60	45	30	90	15	60

Show this information on a pie chart.

17. A farmer grows five types of crops in a 24-hectare farm. The pie chart below shows how the crops are distributed on the farm.



(a) Calculate the area, in hectares, that each type of crop occupies.
(b) Show this information on a bar graph.

18. The mean mass of three men is 62 kg. Four women weigh 61 kg, 66 kg, 65 kg and 69 kg. Find the mean mass of the 7 people.

19. Four farmers have the following number of animals: 25, 23, x and 2x. If the mean number of animals per farmer is 24, find the value of x.

20. A car traveled at a speed of 80 km/h for 30 minutes, 70 km/h for 40 minutes and 90 km/h for 20 minutes. What was the average speed for the whole journey?

21. The average mass of 12 people in a lift is 68 kg. The maximum mass allowed in the lift is 750 kg.
(a) By how many kilograms was the lift overloaded?
(b) How many people should come out of the lift to allow it to move?

22. The table below shows the ages of 11 girls in a soccer team.

Age in years	15	16	17	18	19
Frequency	2	4	2	1	2

Find the:
(a) modal age,
(b) median age,
(c) mean age.

Chapter 3

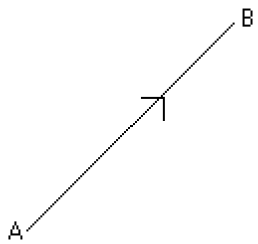
VECTORS

Vector and scalar quantities

When an object is translated, each point of the object moves the same distance and in the same direction. Therefore, we can describe a translation only if we know the distance moved and the direction of motion.
A quantity which describes the distance and direction of a movement is called a **displacement vector** or simply a **vector**.
The distance moved by the object is the length of the vector and it is called the **magnitude** (size) of the vector. Some examples of vectors are velocity, acceleration and force. Some quantities like mass, height, time and temperature have magnitude only. Such quantities are called **scalars**.

Representation of vectors

A vector is represented by a directed line as shown in the figure below.

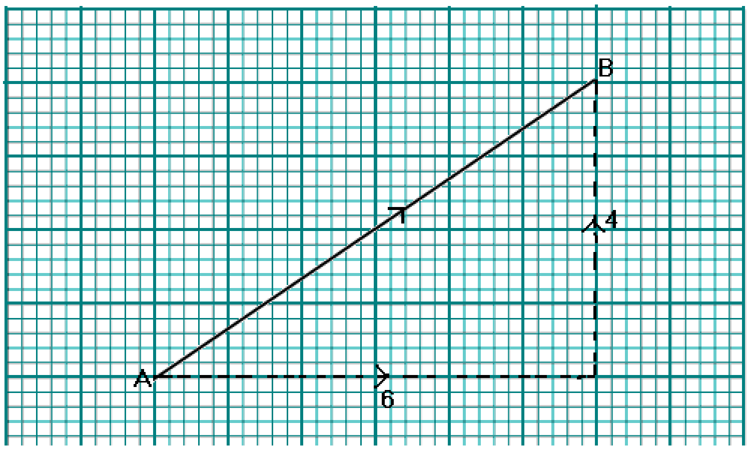


The direction of the vector from A to B is shown by an arrow and its magnitude is the length of the line AB.
Vector AB is written as $\underset{\sim}{AB}$ or $\overrightarrow{AB}$ or **AB.** Its magnitude is denoted by $|AB|$.
A is the initial point (sometimes called the tail) and B is the terminal point (sometimes called the **head**) of the vector.
Sometimes a vector can be denoted by a single small letter as shown in the figure below.

In printed work, vectors are denoted by bold letters, thus vector AB or vector a is written as **AB** or **a**. When writing these vectors you will use the common notation $\underset{\sim}{AB}$ or $\underset{\sim}{a}$.

Translations

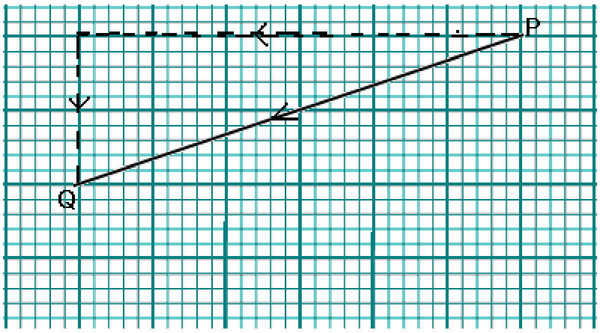
If the object moves in such a way that all its points move through the same distance and in the same direction, the object is said to have undergone a translation.
Consider a translation from point A to B in the figure below.



If each square is of one unit length, the translation is shown by the vector **AB** and can be described as a displacement of 6 units to the right and 4 units upwards. This is written as **AB =** $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$.

When a displacement vector is written in this way it is called a **column vector**. The number at the top represents the horizontal displacement and the lower number represents the vertical displacement.

The figure below shows a displacement from P to Q.



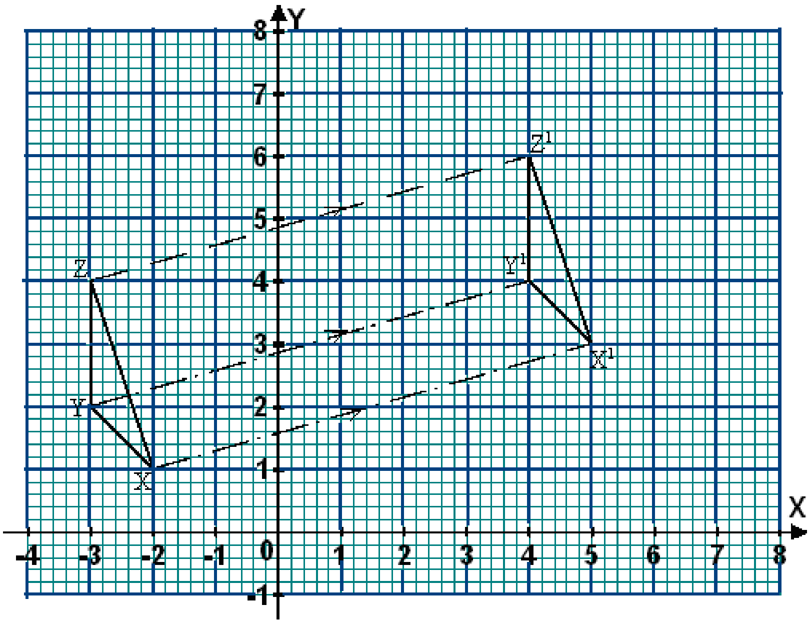
The vector **PQ** is a displacement of 6 units to the left and 2 units downwards. This is written as **PQ** = $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$.

It is important to note that whenever the displacement is towards the right or upwards, it is a positive displacement whilst displacement to the left or downwards is negative.

Example 1

Triangle XYZ is such that X(-2, 1), Y(-3, 2) and Z(-3, 4). It is given a translation of vector $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Find the coordinates of its image, $X^1Y^1Z^1$

Solution:

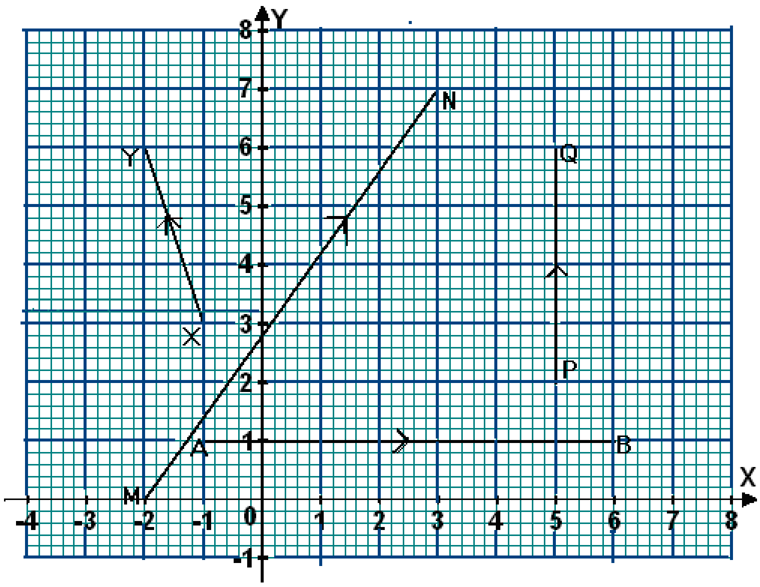


Points X, Y and Z undergo a translation of vector $\begin{pmatrix}7\\2\end{pmatrix}$. The image, $X^1Y^1Z^1$, is such that $X^1(5, 3)$, $Y^1(4, 4)$ and $Z^1(4, 6)$.

Note: *When an object undergoes a translation, all points move through the same distance and in the same direction. Thus, lines XX^1, ZZ^1 and YY^1 are parallel and equal.*

Exercise 3.1

1. Write the column vectors of the translations shown in the figure below.



2. On a squared paper draw the vectors represented by the following column vectors.

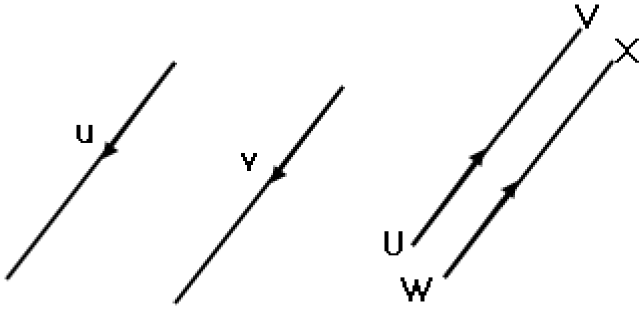
   (a) **PQ** = $\begin{pmatrix}3\\4\end{pmatrix}$ (b) **IJ** = $\begin{pmatrix}-1\\-2\end{pmatrix}$

(c) **RS** = $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ (d) **MN** = $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(e) **AB** = $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ (f) **CD** = $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

Equivalent vectors

Vectors are equal if they have the same magnitude and the same direction. The figure below shows two pairs of equivalent vectors. **UV** = **WX** and **u** = **v.**



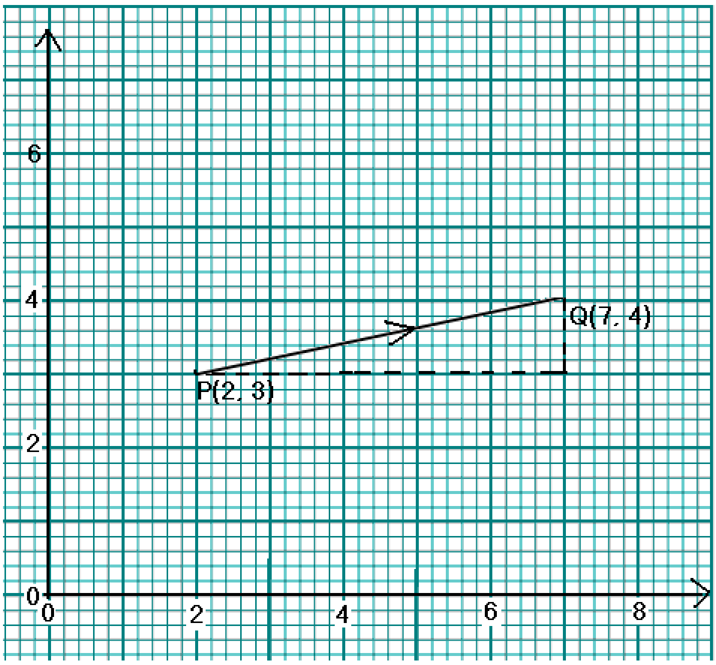
Note: If **AB** = $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then, **BA** = $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$. The vectors **AB** and **BA** have the same length but opposite directions.

Given a point P(x_1, y_1), we can plot the point on a graph paper by locating it x_1 units and y_1 units from the origin. When this point is joined to another point (x_2, y_2), the directed line joining the two points form a vector.

Example 2

Plot the points P(2, 3) and Q(7, 4) and show vector **PQ**.

Solution



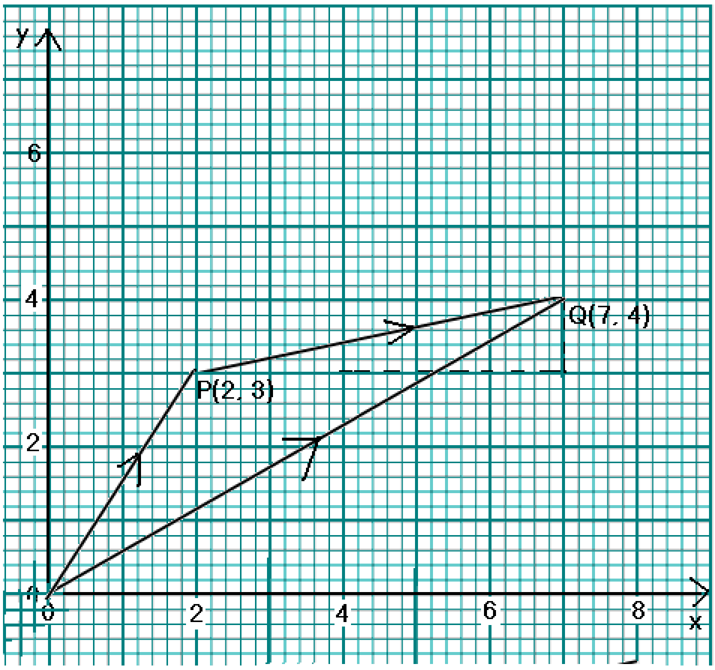
Exercise 3.2

1. A translation maps the origin of the Cartesian plane onto (2, 4). Under the same translation, where would the following points be mapped?
   (a) (-2, 1) (b) (2, 4)
   (c) (-5, -2) (d) (0, 1)

2. A triangle, ABC, is such that A is (1, 1), B(1, 4) and C(5, 1). Find the coordinates of the images of A, B and C when the triangle is translated by vectors:
   (a) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (b) **AB**
   (c) **CA** (d) **CB**

Position vectors

These are column vectors which originate from the origin.



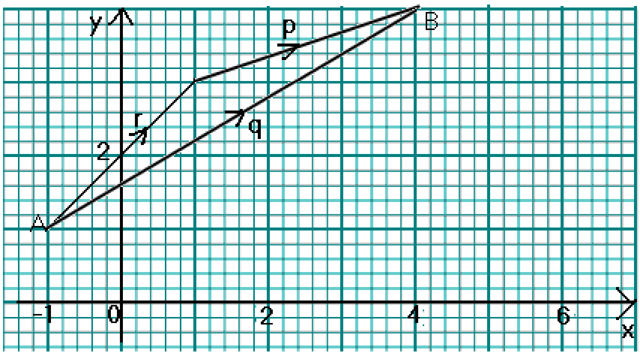
In the figure above, Q is a point whose coordinates are (7, 4). The position vector of R, denoted as **OR** or simply **r**, is given by **OR** = **r** = $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

Also, the coordinates of point P are (2, 3). The position vector of point P, denoted as **OP** or ***p*** is given by **OP** = ***p*** = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

In general, if a point P has coordinates (x, y) then its position vector is
OP = ***p*** = $\begin{pmatrix} x \\ y \end{pmatrix}$.

Adding vectors

An object at point A(-1, 1) is given a translation **r** = $\begin{pmatrix}2\\2\end{pmatrix}$ followed by a translation ***p*** = $\begin{pmatrix}3\\1\end{pmatrix}$ to point B as shown below.



Note that a single translation ***q*** = $\begin{pmatrix}5\\3\end{pmatrix}$ is equivalent to these two translations. We also note that vector ***q*** is the sum of ***r*** and ***p***. Thus,

r+ ***p*** = ***q*** or $\begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2+3\\2+1\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$.

*Although the **r** + **p** is equivalent to **q**, the length of **r** plus the length of **p** is greater than the length of **q**. Therefore, the + sign simply shows that **r** + **p** is a displacement **r** followed by displacement **p**.*

When adding vectors, the horizontal displacements and the vertical displacements are added separately.

Thus, if **a** = $\begin{pmatrix}x_1\\y_1\end{pmatrix}$ and **b** = $\begin{pmatrix}x_2\\y_2\end{pmatrix}$ then,

a + **b** = $\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}$.

Example 3

Given that **a** = $\begin{pmatrix}3\\4\end{pmatrix}$, **b** = $\begin{pmatrix}1\\6\end{pmatrix}$, **c** = $\begin{pmatrix}2\\-3\end{pmatrix}$, find:

(a) **a** + **b** (b) **a** + **c**

(c) **a** + **b** + **c**

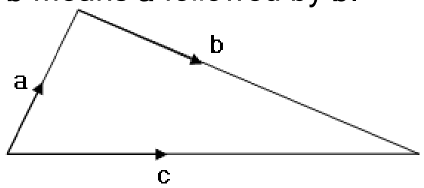
Solutions

(a) **a** + **b** = $\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}1\\6\end{pmatrix} = \begin{pmatrix}3+1\\4+6\end{pmatrix} = \begin{pmatrix}4\\10\end{pmatrix}$.

(b) **a** + **c** = $\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}3+2\\4-3\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix}$.

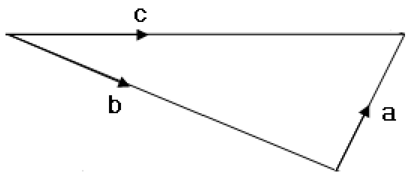
(c) **a** + **b** + **c** = $\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}1\\6\end{pmatrix} + \begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}3+1+2\\4+6-3\end{pmatrix} = \begin{pmatrix}6\\7\end{pmatrix}$.

If a point is given a translation **a** followed by a translation **b**, the resulting displacement is the translation **c**. See figure below. This is written as **a** + **b** = **c**. Where **a** + **b** means **a** followed by **b**.

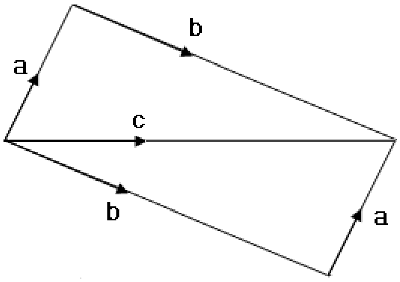


Note that the tail of **b** is joined to the head of **a** so that the tail of the resultant vector, **c**, is joined to the tail of **a**, the initial position, and its head joined to the head of **b**, the final position.

Consider translation **b** followed by translation **a** as in the figure below.



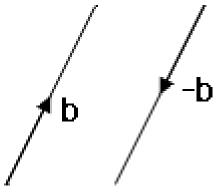
The resulting displacement is still the same translation **c**. Thus **b** + **a** = **c**. It is important to note that the resulting translation can be obtained by starting with either **a** or **b**, as shown in the combined diagram in the figure below.



Note that **a** + **b** = **b** + **a**.

Subtracting vectors

When subtracting, 6 and -4 may be written as 6 + (-4). Similarly, when we subtract vectors, we may write a - b = a + -b. The vector -b is the negative vector of b. Thus, if **b** = $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ then -**b** = $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ and can be represented diagrammatically as shown in the figure below.



Note that both vectors have the same length and are parallel to each other, but are in the opposite directions.

Thus, given **a** = $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and **b** = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$,

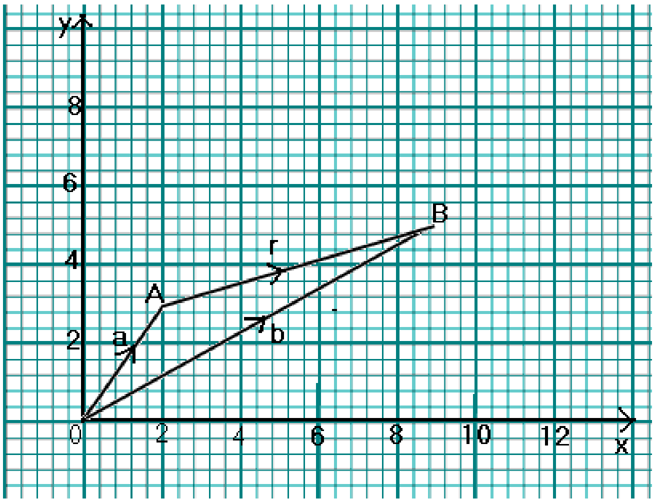
a - **b** = $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ - $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ or **a** - **b** = $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ + $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 7-3 \\ 5-2 \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} 7-3 \\ 5-2 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\qquad$ $= \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Therefore, **a** - **b** = **a** + -**b** = $\begin{pmatrix} 7 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Consider points A(2, 3) and B(9, 5) in the figure below.



The position vectors of A and B are given as

OA = **a** = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and **OB** = **b** = $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$

OA + **AB** = **OB**

AB = **OB** - **OA**

= **b** - **a**

$= \begin{pmatrix} 9 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

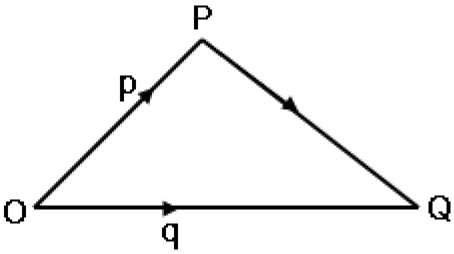
AO = -**OA** = -**a**. Thus, **AB** = **AO** + **OB**.

Example 4

Given that the coordinates of point P are (4, -3) and **PQ** = $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, find the coordinates of point Q.

Solution

The position vector of P is **OP** = $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.



The position vector of Q is given by:
OQ = **OP** + **PQ**

$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

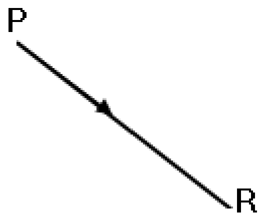
The coordinates of Q are (6, 2).

The zero vector

Consider two vectors, **c** and **d**, such that translation **c** followed by translation **d** returns to the initial point.

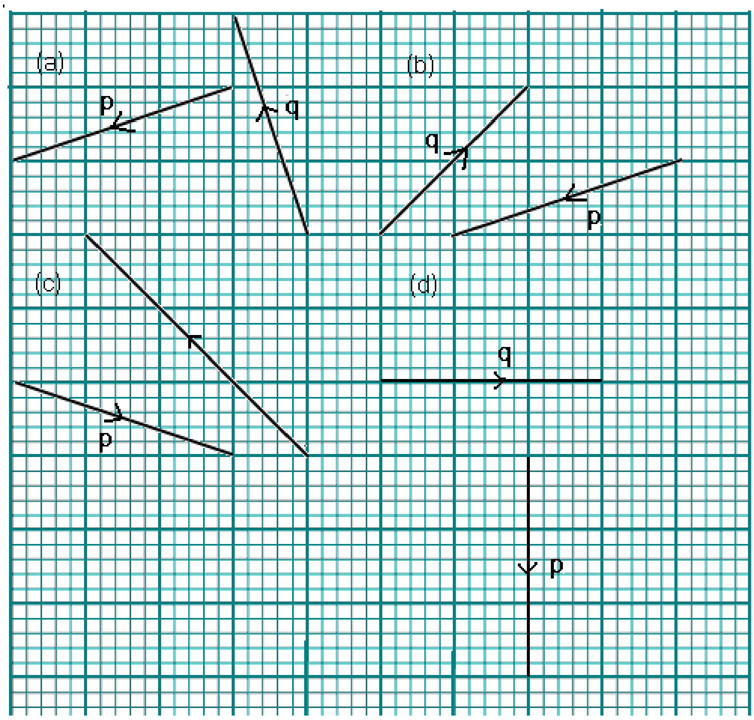
What can you say about the two translations? Do you notice that the vectors of the two translation are exactly the same length but in opposite directions?

Such a result is called a zero or (null) vector. This is because there is effectively no change in the position. A zero vector is written as **O**.Thus,
c + **d** = **O**. Therefore, **c** = -**d**. A vector **RP** has the same magnitude as the vector **PR,** but its direction is opposite to that of **PR**.
PR + **RP** = **O** and **RP** = -**PR**



Exercise 3.3

1. State the position vectors of the following points:
   (a) P(5, 3) (b) Q(2, 3)
   (c) R(-6, 8) (d) S(-3, -4)
   (e) T(0, 2) (f) U(-3, 0)
2. State the coordinates of the points with the following position vectors:
   (a) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$
   (c) $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ (d) $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$
3. In each of the following, draw **p** + **q** and state its column vector.



4. Given that **OR** = **OP** + **OQ**, state the coordinates of R when the coordinates of P and Q are:
   (a) P(0, 1) and Q(3, 6)
   (b) P(-3, 2) and Q(5, 1)
   (c) P(-4, -3) and Q(2, 0)
   (d) P(1, -7) and Q(6, 2).

5. Write down negatives of each of the following vectors:
   (a) $\begin{pmatrix} 7 \\ 11 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$
   (c) $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
   (e) $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$ (f) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6. P is (5, 3), Q is (-4, 2) and R is (2, -3). Find the column vectors **PQ**, **RQ** and **RP.**

7. The coordinates of point A are (2, 1) and **AB** = $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. Find the coordinates of B.

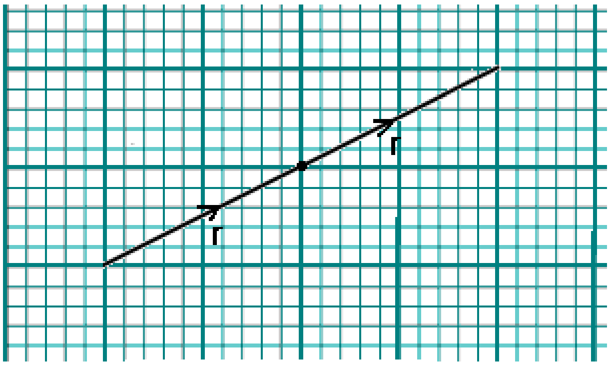
8. Given A(6, 3) and B(-4, 9), find the coordinates of C when:
   (a) **OC** = **OA** + **OB**
   (b) **OC** + **OB** = **OA**

9. Given that **p** = $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$, **q** = $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and **q** + **r** = **p**, express **r** as a column vector.

10. If **r** = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, **s** = $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and **t** = $\begin{pmatrix} -5 \\ -6 \end{pmatrix}$, determine:
   (a) **r + s** (b) **r + s - t** (c) **r - (s + t)**

Multiplying vectors by a scalar

In the figure below, **r** = $\begin{pmatrix}2\\1\end{pmatrix}$ and has been added to itself.



r + **r** = $\begin{pmatrix}2\\1\end{pmatrix}$ + $\begin{pmatrix}2\\1\end{pmatrix}$ = $\begin{pmatrix}4\\2\end{pmatrix}$.

The addition of R to itself gives **r** + **r** = 2**r**. Thus,

2**r** = 2$\begin{pmatrix}2\\1\end{pmatrix}$ = $\begin{pmatrix}2\times 2\\2\times 1\end{pmatrix}$ = $\begin{pmatrix}4\\2\end{pmatrix}$. The number 2 is a scalar value in 2**r**. In general, when a vector $\begin{pmatrix}a\\b\end{pmatrix}$ is multiplied by a scalar k, we get $k\begin{pmatrix}a\\b\end{pmatrix}$ = $\begin{pmatrix}ka\\kb\end{pmatrix}$. The scalar k can be any positive or negative number. Each component of the vector is multiplied by the scalar.

Example 5

Given **a** = $\begin{pmatrix}2\\3\end{pmatrix}$ and **b** = $\begin{pmatrix}-4\\10\end{pmatrix}$, find the value of:

(a) 4**a**

(b) $-\frac{1}{2}$**b**

(c) 3**a** - $\frac{1}{2}$**b**

(d) 3**a** + 2**b**

Solution

(a) 4**a** = 4$\begin{pmatrix}2\\3\end{pmatrix}$ = $\begin{pmatrix}8\\12\end{pmatrix}$

(b) $-\frac{1}{2}$**b** = $-\frac{1}{2}\begin{pmatrix}-4\\10\end{pmatrix}$ = $\begin{pmatrix}-2\\5\end{pmatrix}$

(c) 3**a** - $\frac{1}{2}$**b** = 3$\begin{pmatrix}2\\3\end{pmatrix}$ $-\frac{1}{2}\begin{pmatrix}-4\\10\end{pmatrix}$ = $\begin{pmatrix}6\\9\end{pmatrix}$ + $\begin{pmatrix}2\\-5\end{pmatrix}$

= $\begin{pmatrix}8\\4\end{pmatrix}$.

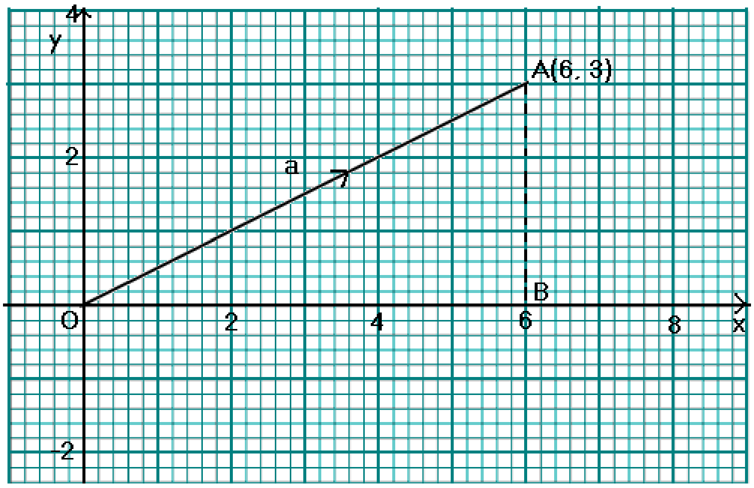
(d) 3**a** + 2**b** = 3$\begin{pmatrix}2\\3\end{pmatrix}$ + 2$\begin{pmatrix}-4\\10\end{pmatrix}$

= $\begin{pmatrix}6\\9\end{pmatrix}$ + $\begin{pmatrix}-8\\20\end{pmatrix}$ = $\begin{pmatrix}-2\\29\end{pmatrix}$

Note: *When a vector is multiplied by a positive scalar, its direction does not change. However, when a vector is multiplied by a negative scalar, its direction is reversed.*

The magnitude of a vector

The length of a vector is called **magnitude**. In the figure below the magnitude of **OA** is the length of line OA and is denoted as $|\mathbf{OA}|$ or $|\mathbf{a}|$ or a.



Triangle OAB is right-angled at point B. We can use the Pythagoras' theorem to find the length of line OA. Thus

$$\overline{OA}^2 = \overline{OB}^2 + \overline{BA}^2$$
$$= 6^2 + 3^2$$
$$= 36 + 9$$
$$= 45$$
$$\overline{OA} = \sqrt{45}$$

Therefore, $|OA| = |a| = 6.708$ units.

Note that the column vector of **OA** is $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $|OA|$ is $\sqrt{6^2 + 3^2}$.

In general, the magnitude of a column vector **t =** $\begin{pmatrix} a \\ b \end{pmatrix}$ also denoted as |**t**|, is given by

$\sqrt{a^2 + b^2}$

Since the magnitude of a vector is its length, the quantity is always a positive scalar.

Example 6

Given that, **a** = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and **b** = $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$ determine:

(a) |**a**| (b) |**b**|

(c) |**a** + **b**| (d) |**a** - **b**|

Solutions

(a) |**a**| = $\sqrt{2^2 + 3^2}$ = $\sqrt{13}$ = 3.606

(b) $|\mathbf{b}| = \sqrt{5^2 + (-4)^2} = \sqrt{25+16}$

$= \sqrt{41}$

$= 4.403.$

(c) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$|\mathbf{a} + \mathbf{b}| = \sqrt{7^2 + (-1)^2} = \sqrt{50}$

$= 7.071.$

(d) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

$|\mathbf{a} - \mathbf{b}| = \sqrt{(-3)^2 + 7^2} = \sqrt{9+49}$

$= \sqrt{58} = 7.616$

Example 7

Given $\mathbf{r} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find:

(a) $|\mathbf{r}|$

(b) $|2\mathbf{r}|$

(c) $2|\mathbf{r}|$

(d) $|k\mathbf{r}|$

Solutions

(a) $|\mathbf{r}| = \sqrt{3^2 + (-4)^2}$

$= \sqrt{9+16} = \sqrt{25} = 5.$

(b) $2\mathbf{r} = 2\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$

$|2\mathbf{r}| = \sqrt{6^2 + (-8)^2}$

$= \sqrt{36+64}$

$= \sqrt{100} = 10$

(c) $2|\mathbf{r}| = 2\left(\sqrt{3^2 + (-4)^2}\right)$

$= 2\sqrt{25} = 2 \times 5 = 10$

(d) $k\mathbf{r} = k\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3k \\ -4k \end{pmatrix}$

$|k\mathbf{r}| = \sqrt{(3k)^2 + (-4k)^2}$

$= \sqrt{9k^2 + 16k^2}$

$= \sqrt{25k^2} = 5k$

From this example, multiplying a vector by a scalar, k, also multiplies its magnitude by k. In general, $|k\mathbf{r}| = k|\mathbf{r}|$.

Exercise 3.4

1. Evaluate the following:

(a) $2\begin{pmatrix}3\\1\end{pmatrix}$ (b) $10\begin{pmatrix}\frac{1}{2}\\\frac{1}{5}\end{pmatrix}$

(c) $5\begin{pmatrix}-4\\-5\end{pmatrix}$ (d) $3\begin{pmatrix}0\\1\end{pmatrix}$

(e) $-8\begin{pmatrix}-1\\-1\end{pmatrix}$ (f) $-7\begin{pmatrix}-1\\0\end{pmatrix}$

(g) $\frac{1}{3}\begin{pmatrix}3\\9\end{pmatrix}$ (h) $\frac{-1}{5}\begin{pmatrix}-10\\-5\end{pmatrix}$

2. Given A(6, 3) and B(-4, 12), find the coordinates of C when:

(a) **OC** = $\frac{1}{3}$**OA** + $\frac{1}{4}$**OB**

(b) **OC** = 2**OA** + **OB**,

(c) **OC** + $\frac{1}{2}$**OB** = 2**OA**

3. If **r** = $\begin{pmatrix}1\\3\end{pmatrix}$, **s** = $\begin{pmatrix}3\\-4\end{pmatrix}$ and **t** = $\begin{pmatrix}1\\3\end{pmatrix}$, determine:

(a) 4**r** - **t** (b) 2**s** - 3**r**

(c) 3**r** + **s** + $\frac{1}{3}$**t**

4. Determine the magnitudes of the following vectors:

(a) $\begin{pmatrix}4\\2\end{pmatrix}$ (b) $\begin{pmatrix}1\\3\end{pmatrix}$

(c) $\begin{pmatrix}-6\\8\end{pmatrix}$ (d) $\begin{pmatrix}-4\\-7\end{pmatrix}$

5. Given that **a** = $\begin{pmatrix}5\\3\end{pmatrix}$ and **b** = $\begin{pmatrix}-8\\6\end{pmatrix}$ calculate:

(a) $|\frac{1}{2}\mathbf{b}|$ (b) $|\mathbf{a} - \frac{1}{2}\mathbf{b}|$

(c) |**a** + **b**| (d) |-2**a**|

6. Given that **r** = $\begin{pmatrix}a\\-3\end{pmatrix}$ and **s** = $\begin{pmatrix}a-1\\2\end{pmatrix}$ and |**r**| = |**s**|, find a.

7. If **a** = $\begin{pmatrix}1\\y-1\end{pmatrix}$, **b** = $\begin{pmatrix}3\\y+2\end{pmatrix}$ and |**a**| = |**b**|, find the value of y.

8. PQR is a straight line such that $\overline{PQ} = 2\overline{QR}$.

(a) Given P(6, 0) and R(4, 3), write down the column vectors for **OP** and **OR**.

(b) If **OP** = **p** and **OR** = **r**, express **RP**, **RQ** and **OQ** in terms of **p** and **r**. Hence find the coordinates of point Q.

Equality of vectors

Two vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are equal if $x_1 = x_2$ and $y_1 = y_2$.

Note: Two vectors are equal if they have the same magnitude and same direction

Example 8

Given **a** = $\begin{pmatrix} 6k \\ 4 \end{pmatrix}$, **b** = $\begin{pmatrix} 10 \\ 5+n \end{pmatrix}$ and **a** = **b**, determine the value of k and n.

Solution

$\begin{pmatrix} 6k \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 5+n \end{pmatrix}$, then 6k = 10 or k = $\frac{10}{6} = \frac{5}{3}$ and 4 = 5 + n. Therefore, n = -1

k = $\frac{5}{3}$ and n = -1.

The null vector

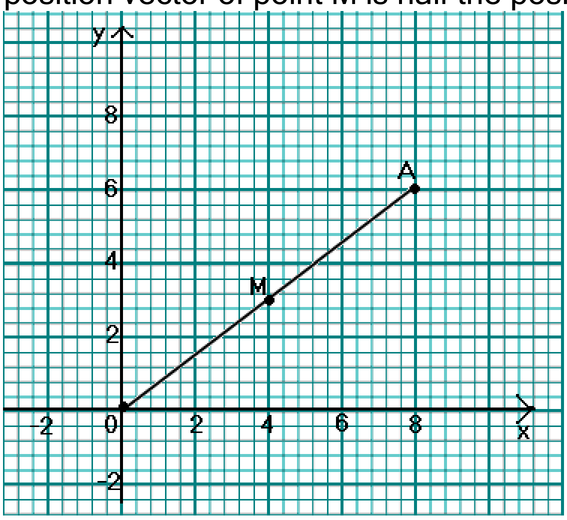
The vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has no magnitude and no direction. It is therefore, a **null** or **zero** vector dented as **O** or $\vec{O}$

Mid-points

A point that bisects a vector is called its mid-point. To find the mid-point of a vector we get the point that lies halfway on the position vector.
Suppose M is the mid-point of **OA** in the figure below, then

OA = $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ and **OM** = $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. **OM** = $\frac{1}{2}$**OA**.

This means that the position vector of point M is half the position vector of point A.



If M is the mid-point of a line AB where A is the point (x_1, y_1) an B is the point (x_2, y_2) then the coordinates of M are given by
M = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 9

Find the mid-point, N, of points A(1, 5) and B(3, 7).

Solution

$x_1 = 1$, $y_1 = 5$; $x_2 = 3$, $y_2 = 7$

The coordinates of N are $\left(\frac{1+3}{2}, \frac{5+7}{2}\right)$

Therefore, N$\left(\frac{4}{2}, \frac{12}{2}\right)$ = N(2, 6).

Exercise 3.5

1. PQRS is a parallelogram with points P3, 1), Q(12, 5) and R(6, 8). Find:
   (a) **PQ**,
   (b) the coordinates of S
   (c) |**QS**|.

2. Given A(1, 4), B(6, 7) and D(1, -2), and that ABCD is a trapezium such that **DC** = 3**AB**, find the coordinates of C.

3. Show that the points U(-4, -2), V(-1, -3), W(5, 0) and X(2, 1) are vertices of a parallelogram.

4. Find the mid-point of **AB** given A(-8, 5) and B(1, 4).

5. Given X(7, 14), **XY** = $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$. Find:
   (a) the coordinates of Y,
   (b) **XN**,
   (c) the coordinates of N

6. The point M$\left(\frac{1}{2}, -1\right)$ is the mid-point of points A(a, -3) and B(4, b). Find a and b.

7. Triangle ABC with vertices A(-1, 1), B(1, 3) and C(2, 1) is translated by vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find the coordinates of the vertices of the image.

8. Given that **a** = $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and **b** = $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$, find:
   (a) **a** + 2**b** (b) 3**a** - $\frac{1}{2}$**b**
   (c) |**a** - 2**b**|

9. Given **r** = **a** - 2**b** and **s** = 3**a** + **b**, express the following vectors in terms of **a** and **b**:
   (a) **r** - **s** (b) 4**s** + 3**r**

10. Given point B(5,3) and **AB** = $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find the coordinates of point A.

11. Given point N(-6, 8) and **NM** = $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$, find the mid-point of NM.

12. Show that quadrilateral ABCD with vertices A(1, 2), B(4, 4), C(4, 1) and D(1, -1) is a parallelogram. Determine the coordinates of the point of intersection of its diagonals.

13. Point M(-3, 4) is the mid-point of points A and B. If the coordinates of A are (-5, 1), find the coordinates of B.

14. Given that quadrilateral PQRS with vertices P(2, 4), Q(8, 8), R(8, 2) and S(2, -2) is a parallelogram. Find:
    (a) the magnitudes of the diagonals,
    (b) the coordinates of the point of intersection of its diagonals.

Chapter 4

COMPUTATION

Squares of numbers

When a number is multiplied by itself, the product is called its square. For example, $9 \times 9 = 81$; 81 is the square of 9. In short the square of 9 is written as 9^2. Thus, $9^2 = 81$.

Square tables

Tables of squares are normally provided together with other mathematical tables, to enable us do quick calculations. In this section you will learn how to find squares of whole numbers and decimals using tables. In the tables, squares of numbers between 1.000 and 9.999 are given. Table 1 below is an extract from the table of squares.

The table has three main sections as follows:

- Column headed x, featuring from top to bottom the numbers 1.0 - 9.9
- Ten main columns headed 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.
- Mean difference columns headed 1 to 9.

Example 1

Use square tables to find the values of:
(a) 1.2^2 (b) 1.28^2

Solution

(a) 1.2^2

- Move down the column headed x and read 1.2
- Move along this row to the column headed 0. The number at the intersection of this row and the column for 0 is 1.440.
- 1.440 is the square of 1.2.

$1.2^2 = 1.440$.

Note: Square tables give values up to 4 significant figures.

(b) 1.28^2

- Move down the column headed x and read 1.2.
- Move along the row starting from 1.2 until you reach the column headed 8. The number at the intersection of this row and the column is 1.638. Therefore, $1.28^2 = 1.638$.

An extract from the tables of squares

SQUARES x^2											ADD								
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	10	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	11	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	2	5	7	10	12	15	17	20	22
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	13	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	14	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	15	19	22	25	28

Example 2

Use the above table to find the values of:

(a) 14.6^2 (b) 1.463^2

Solutions

(a) Note that 14.6 is not a number between 1.000 and 9.999 and therefore it cannot be read from the tables directly. We can change it to a number between 1.000 and 9.999 using the standard form.

$14.6 = 1.46 \times 10$. Therefore, $14.6^2 = 1.46^2 \times 10^2$ or $1.46^2 \times 100$.

We can get the value of 1.46^2 from tables.

- Move down the column marked x and read 1.4.
- Along the row marked 1.4 move until you reach the column marked 6. The number at the intersection of this row and the column is 2.132

Therefore, $1.46^2 = 2.132$

$14.6^2 = 1.46^2 \times 100$

$\therefore 14.6^2 = 2.132 \times 100$

$= 213.2$

(b) The number 1.463^2 has four figures. To find its square, the *mean difference column* (ADD section) is used.

- 1.4 is read in the column marked x.
- From 1.4 move along the row to the column headed 6. The number at this point is 2.132.
- Move ahead along the same row to the section headed ***ADD***. Under the column headed 3, the number at the point of intersection is 9.
- Add this number to the figure 2.132 obtained as the square of 1.46. That is, 2.132

$$\begin{array}{r} 2.132 \\ +\quad 9 \\ \hline 2.141 \\ \hline \end{array}$$

$\therefore 1.463^2 = 2.141$

Normally, positive numbers greater than 10 and less than 1 are expressed in **standard form** before using the square tables. To express a number in the form $A \times 10^n$ where $1 \leq A \leq 10$, and n is an integer is to express the number in standard form.

Example 3

Find the values of the following using tables.

(a) 25.3^2 (b) 223.4^2

Solution

(a) Express 25.3 in standard form.

$25.3 = 2.53 \times 10$

$\therefore 25.3^2 = (2.53 \times 10)^2$

Using tables:

$2.53^2 = 6.401$

$\therefore 25.3^2 = 2.53^2 \times 10^2$

$= 6.401 \times 100$

$= 640.1$

(b) Express 223.4 in standard form:

$223.4 = 2.234 \times 10^2$

$\therefore 223.4^2 = 2.234^2 \times (10^2)^2$

Using tables:

$2.234^2 = 4.973$

$+ \quad 18$

4.991

$\therefore 223.4^2 = 2.234^2 \times (10^2)^2$

$= 4.991 \times 10\,000$

$= 49\,910.$

Exercise 4.1

Find the squares of the following numbers.

1. 2.7
2. 3.85
3. 6.123
4. 38.9
5. 86.7
6. 79.86
7. 83.57
8. 496.8
9. 263.4
10. 1639
11. 14.24
12. 2.002

Squares of numbers less than 1

Numbers less than 1 can be expressed in standard form. For example:

$0.367 = 3.67 \times 10^{-1}$

$0.0367 = 3.67 \times 10^{-2}$

$0.00367 = 3.67 \times 10^{-3}$

When the power of 10 is a negative integer, this is the same as dividing by 10 raised to the power of a positive integer. For example:

$3.67 \times 10^{-1} = 3.67 \div 10^1$

$3.67 \times 10^{-2} = 3.67 \div 10^2$

$3.67 \times 10^{-3} = 3.67 \div 10^3$

Example 4

Use tables to find the squares of:

(a) 0.43 (b) 0.0642

Solution

(a) Express 0.43 in standard form

$0.43 = 4.3 \times 10^{-1}$

$= 4.3 \div 10^{1}$

$\therefore (0.43)^2 = 4.3^2 \div (10^1)^2$

$= 18.49 \div 10^2$

$= 0.1849$

(b) Express 0.0642 in standard form.

$0.0642 = 6.42 \times 10^{-2}$

$\therefore 0.0642^2 = 6.42^2 \times (10^{-2})^2$

$= 41.22 \times 10^{-4}$

$= 41.22 \div 10000$

$= 0.004122$

Exercise 4.2

Use square tables to find the values of:

1. 0.16^2
2. 0.38^2
3. 0.056^2
4. 0.049^2
5. 0.0063^2
6. 0.213^2
7. 0.00364^2
8. 0.00042^2
9. 0.8^2
10. 0.082^2
11. 0.0674^2
12. 0.00492^2
13. 0.0649^2
14. 0.0021^2
15. 0.625^2
16. Evaluate:

(a) $93^2 + 52^2$

(b) $87^2 - 43.4^2$

(b) $161.2^2 + 19.4^2$

(c) $0.239^2 + 0.6612^2$

Square roots

The square of 4 is 16. The square root of 16 is 4. The sign for the square root is $\sqrt{\ }$.

Thus $\sqrt{16}$ = 4 and $\sqrt{36}$ = 6.

Finding square roots by factorization

Factorization can be used to find the square root of a number that can be expressed in terms of its prime factors. The factor should be in pairs or in powers of even numbers.

Example 5

Find the square roots of the following numbers:

(a) 1225

(b) 1.44

Solution

(a) $1225 = 5 \times 245$

$= 5 \times 5 \times 49$

$= 5 \times 5 \times 7 \times 7$

$= 5^2 \times 7^2$

$\sqrt{1225} = \sqrt{5^2} \times \sqrt{7^2}$

$= 5 \times 7$

$= 35$

(b) $1.44 = \frac{144}{100}$

$= \frac{12 \times 12}{10 \times 10} = \frac{12^2}{10^2}$

$\sqrt{1.44} = \sqrt{\frac{12^2}{10^2}} = \frac{12}{10} = 1.2$

Alternatively,

$144 = 2 \times 72$

$= 2 \times 2 \times 36$

$= 2 \times 2 \times 2 \times 18$

$= 2 \times 2 \times 2 \times 2 \times 9$

$= 2 \times 2 \times 2 \times 2 \times 3 \times 3$

$= 2^4 \times 3^2$

$\therefore \sqrt{1.44} = \frac{\sqrt{2^4 \times 3^2}}{\sqrt{2^2 \times 5^2}}$

$= \frac{2^2 \times 3}{2 \times 5}$

$= \frac{12}{10} = 1.2$

Exercise 4.3

Find, by factorization the square roots of:

1. 441
2. 3249
3. 3.625
4. 7569
5. 99225
6. 2.56
7. 23.04
8. 0.7056
9. 0.001369
10. 0.05
11. 400
12. 0.0004

Square root tables

A faster way of finding the square root of a number is by use of square root tables. The square root tables give the square roots of numbers from 1.000 to 9.999 and from 10.00 to 99.99. Table 4.1 (a) is an extract from square root tables for value between 1.000 and 9.99. The square root tables have three sections similar to those of the square tables and these sections are used in a similar way.

Table 4.1 (a)

SQUARE ROOTS from 1 to 10											ADD								
x	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	0	1	1	1	2	2	2	3	3
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	0	1	1	1	2	2	2	3	3
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	0	1	1	1	2	2	2	3	3
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	0	1	1	1	2	2	2	3	3
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	0	1	1	1	2	2	2	3	3
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	0	1	1	1	2	2	2	3	3
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	0	1	1	1	2	2	2	2	3
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	0	1	1	1	2	2	2	2	3
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	0	1	1	1	1	2	2	2	3
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	0	1	1	1	1	2	2	2	3

Example 6

Find the square root of 2.65.

Solution

The square root of 2.65 will be read from the tables with numbers between 1.000 and 9.999. The square root is obtained in the same way the squares are read from the square tables.

$\sqrt{2.65}$ = 1.628

Square roots of numbers less than 1 and greater than 100.

To find the square root of a number that is less than 1 or greater than 100, the number should be expressed in a form that lies between 1 and 100. For example, $225 = 22.5 \times 10$ or 2.25×10^2.

Note that we take the form in which 10 is to the power of an even number.

Thus, $\sqrt{225} = \sqrt{2.25} \times \sqrt{10^2}$

$= 1.5 \times 10 = 15.$

Example 7

Find the square root of:

(a) 0.235 (b) 0.0168

(c) 4863

Solution

(a) $0.235 = 23.5 \times 10^{-2}$

$\therefore \sqrt{0.235} = \sqrt{23.5} \times \sqrt{10^{-2}}$

$= 4.848 \times 10^{-1}$

$= 4.848 \div 10$

$= 0.4848$

(b) $0.0168 = 1.68 \times 10^{-2}$

$\sqrt{0.0168} = \sqrt{1.68} \times \sqrt{10^{-2}}$

$= 1.296 \times 10^{-1}$
$= 1.296 \div 10$
$= 0.1296$

(c) $4863 = 48.63 \times 10^2$

$\sqrt{4863} = \sqrt{48.63} \times \sqrt{10^2}$
$= 6.973 \times 10$
$= 69.73$

Exercise 4.4

Find the square roots of the following from the tables:

1.	8	**2.**	7
3.	13	**4.**	77
5.	69.5	**6.**	8.72
7.	10.5	**8.**	0.97
9.	0.3475	**10.**	215
11.	4267	**12.**	0.018
13.	0.0273	**14.**	0.008471
15.	0.0006477	**16.**	0.0007982
17.	87273	**18.**	0.10758
19.	234000	**20.**	99999

21. Evaluate each of the following:
(a) $\sqrt{0.0792}$ (b) $\sqrt{0.00256}$
(c) $\sqrt{563}$ (d) $\sqrt{0.364}$

22. Work out the following using tables:
(a) 2.5×250 (b) 6.8×68
(c) 16.3×1630 (d) 352×35.2

23. Evaluate without using tables:
(a) $\dfrac{\sqrt{0.81} \times \sqrt{0.64}}{\sqrt{0.36}}$ (b) $\dfrac{\sqrt{0.0144} \times \sqrt{0.625}}{\sqrt{0.04} \times \sqrt{0.25}}$

Reciprocals of numbers

Reciprocals of numbers by division.

Given a number x, its reciprocal is $\frac{1}{x}$. For instance, if x = 2, its reciprocal is $\frac{1}{2}$. Also when x = $\frac{7}{2}$ or 3.5, its reciprocal is $\frac{2}{7}$ or $\frac{1}{3.5}$. We can use long division to find the values of reciprocals.

Example 8

Find the value of the reciprocals of each of the following numbers correct to 4 significant figures:

(a) 15 (b) 1.45

(c) 0.245 (d) $\frac{3}{7}$

Solutions

(a) The reciprocal of 15 = $\frac{1}{15}$. To find the value of the reciprocal we use the long division method.

$$\begin{array}{r|l} & 0.06666.. \\ 15 & 1.00 \\ & -90 \\ & 100 \\ & -90 \\ & 100 \\ & -90 \\ & 100 \\ & -90 \end{array}$$

Thus, $\frac{1}{15} \approx 0.06667$

Using a similar method, we can show that:

(b) $\frac{1}{1.45} \approx 0.6897$

(c) $\frac{1}{0.245} \approx 4.082$

(d) $\frac{1}{\frac{3}{7}} = \frac{7}{3} \approx 2.333$

Table of reciprocals

Finding reciprocals by division is a very tedious process. A quicker way is to use the table of reciprocals. The table of reciprocals provides reciprocals of numbers from 1 up to numbers less than 10. However, if the number x is less than 1 or greater than 10, it is first expressed in standard form. The table of reciprocals is read in the same way as the table of square roots or the table of squares.

Example 9

Use the table of reciprocals to find the reciprocals of each of the following numbers:

(a) 7 (b) 8.23

(c) 1.752 (d) 4.0126

Solutions

(a) In the column marked x, find the row marked 7.0. Move to the right along this row up to the column marked 0. Read the number at the intersection of this row (7.0) and the column 0. It is 0.1429. This is the reciprocal of 7.0. Thus, $\frac{1}{7.0}$ = 0.1429

(b) In the column headed x, find the row marked 8.2. Move along this row up to the column headed 3. Read the number at the intersection of the row and the column. The number is 0.1215. Thus, $\frac{1}{8.23}$

(c) In the column headed x, find the row marked 1.7. Move along this row up to the main column headed 5. Read the number at the intersection of the row and the column. The number is 0.5714. Move further along this row until the column headed 2 in the differences column. Read the number at the intersection of this column and the row. The number is 7. We write this number as 0.0007. Subtract 0.0007 from 0.5714 to get 0.5707. Thus,

$$\frac{1}{1.752} \approx 0.5707$$

(d) Round off 4.0126 correct to 4 significant figures to get 4.013. Then following a similar procedure as above in (c), we get

$$\frac{1}{4.0126} \approx 0.2491$$

Example 10

Find the reciprocal of the following numbers:

(a) 231 (b) 0.00463

Solutions

(a) $\frac{1}{231} = \frac{1}{2.31 \times 10^2} = \frac{1}{2.31} \times \frac{1}{10^2}$

$= 0.4329 \times 10^{-2}$

$= 4.329 \times 10^{-1} \times 10^{-2}$

$= 4.329 \times 10^{-3}$

$= 0.004329$

(b) $\frac{1}{0.00463} = \frac{1}{4.63 \times 10{-3}}$

$= \frac{1}{4.63} \times \frac{1}{10{-3}}$

$= 0.2160 \times 10^3$

$= 2.160 \times 10^2 = 216.0$

Exercise 4.5

1. Use the table of reciprocals to find the values of the following correct to 4 s.f:

(a) $\frac{1}{17}$ (b) $\frac{1}{35.8}$

(c) $\frac{1}{128.5}$ (d) $\frac{1}{0.186}$

(e) $\frac{1}{0.098}$ (f) $\frac{1}{0.007183}$

2. Find, correct to 3 s.f., the values of:

(a) $\frac{1}{\sqrt{5} + \sqrt{68}}$ (b) $\frac{1}{\sqrt{0.83} + 0.46}$

3. Given that $\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$, where u = 25 cm and f = 10 cm. Find v correct to 2 decimal places.

4. One sterling pound is equivalent to 1.592 US dollars. How many sterling pounds are equal to one dollar?

5. Calculate the value of r, correct to 4 s.f., given that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, when $r_1 = 25$, $r_2 = 18$ and $r_3 = 37$.

Chapter 5.

RATIONAL NUMBERS AND POWERS

Rational numbers and Irrational numbers

A number which is either an integer or can be expressed as a fraction of two integers (i.e. in the form $\frac{p}{q}$, where p and q are integers having no common divisor and $q \neq 0$) is called a **rational number**. For example $\frac{3}{4}$, $\frac{2}{7}$, -1, -2, $\frac{5}{9}$ are all rational numbers.

Terminating and Recurring Decimals

Some fractional numbers can be expressed as terminating decimals. For example,
$\frac{3}{4} = 0.75$; $\frac{1}{4} = 0.25$; $\frac{51}{100} = 0.51$.
Conversely, terminating decimals can easily be expressed as a ratio of two integers and therefore are rational numbers.
When we change $\frac{2}{3}$ to a decimal, the digit 6 repeats itself continuously without an end. We say that digit 6 is recurring. Decimals with such recurring digits are called **recurring decimals**.
In contrast, 0.235294… is neither recurring nor a terminating decimal. It is a non-terminating and non-recurring decimal.
Recurring decimals can have one digit which recurs or a group of digits that recur. For example, 0.3333.. has one recurring digit. 0.9595… has 2 recurring digits. A dot is put on top of the recurring digit when there is only one recurring digit. In the case of two or more recurring digits, the dots are put on top of the first and the last recurring digits. For example,

$0.9595\ldots = 0.\dot{9}\dot{5}$

$0.1245245\ldots$ has 3 recurring digits, 245. Therefore, $0.1245245\ldots = 0.1\dot{2}4\dot{5}$

Recurring decimals to fractions

Example 1

Express the following recurring decimals as fractions.

(a) $0.\dot{5}$ (b) $0.2\dot{3}$ (c) $0.\dot{4}6\dot{8}$ (d) $0.\dot{8}\dot{1}81$

Solutions

(a) Let n stand for any decimal number.

n = 0.5555……………………………(i)
Multiply both sides by 10 to get:
10n = 5.555……………………………(ii)

Subtracting (i) from (ii) gives:
10n - n = 5.555… - 0.555… = 5.000
$9n = 5$
$n = \frac{5}{9}$

$\therefore\ 0.\dot{5} = \frac{5}{9}$

(b) n = 0.2333...................... (i)
Multiply both sides of the equation by 10 to get:

10n = 2.3333.......................(ii)

Note that after multiplying by 10, the decimal parts of equations (i) and (ii) are not the same. Multiply equation (ii) by 10 to get:
100n = 23.333 ....(iii)
Now the decimal parts of equations (ii) and (iii) are the same.
Subtracting (ii) from (iii) gives:

$$\begin{array}{rl} 100n - 10n = & 23.333... \\ & -2.333... \\ \hline & 21 \end{array}$$

90n = 21

$n = \frac{21}{90} = \frac{7}{30}$

$\therefore\ 0.2\dot{3} = \frac{7}{30}$.

Note that subtraction should only be done after the decimal number has been multiplied by the appropriate powers of 10 (i.e. 10, 100, 1000, etc.) so that there are only the recurring digits after the decimal point.

(c) n = 0.468468..............................(i)
1000n = 468.468..................................(ii)
Subtracting (i) from (ii) gives:

$$\begin{array}{rl} 1000n - n = & 468.468... \\ & -0.468... \\ \hline & 468 \end{array}$$

999n = 468

$n = \frac{468}{999} = \frac{52}{111}$

(d) n = 1.8181...(i)
100n = 181.8181.......................................(ii)
Subtracting (i) from (ii) gives:

$$\begin{array}{rl} 100n - n = & 181.8181... \\ & -\ 1.8181... \\ \hline = & 180 \end{array}$$

99n = 180

$n = \frac{180}{99} = \frac{20}{11} = 1\frac{9}{11}$.

Exercise 5.1

Express the following fractions as decimals.

1. $\frac{1}{3}$ 2. $\frac{8}{11}$ 3. $\frac{13}{30}$ 4. $\frac{5}{27}$

5. $\frac{15}{37}$ 6. $\frac{1}{27}$ 7. $5\frac{2}{3}$ 8. $1\frac{2}{9}$

Express the following as fractions in their lowest terms.

9. $0.\dot{0}\dot{9}$ 10. $0.1\dot{6}$ 11. $0.\dot{5}2\dot{2}$

12. $1.\dot{4}\dot{5}$ 13. $0.\dot{8}$ 14. $3.\dot{4}$

15. $1.6\dot{3}$ 16. $3.1\dot{2}$ 17. $0.5\dot{6}7\dot{1}$

18. $0.\dot{3}1\dot{5}$ 19. $2.\dot{7}\dot{2}$ 20 $3.\dot{7}\dot{1}$

Standard form

When a number is expressed as $A \times 10^n$ where $1 \leq A < 10$, that is A is equal to or greater than 1 but less than 10 and n is an integer, the number is said to be in **standard form**.

For example:

$$49 = 4.9 \times 10^1 \; (10^1 = 10)$$

$$596 = 5.96 \times 10^2 \; (10^2 = 100)$$

$$1274.3 = 1.2743 \times 10^3 \; (10^3 = 1000)$$

It is assumed that every whole number has a decimal point after the ones digit. You will notice that the decimal point is moved to the left so as to obtain a number between 1 and 10. The power of 10 is equal to the number of places moved to the left. In this case, the power of 10 is a positive integer.

If the decimal point is moved to the right so as to obtain a number between 1 and 10, the power of 10 will also be equal to the number of decimal places moved. However, in this case the power of 10 will be a negative integer. For example:

$$0.359 = 3.59 \times 10^{-1} \; (10^{-1} = \frac{1}{10})$$

$$0.0462 = 4.62 \times 10^{-2} \; (10^{-2} = \frac{1}{100})$$

$$0.003981 = 3.981 \times 10^{-3} \; (10^{-3} = \frac{1}{1000})$$

Numbers in standard form are convenient to work with. Scientists sometimes deal with very large and, at other times, very small numbers of units of measurements. For example, the distance between the earth and the sun is approximately 150,000,000 km. This measurement is expressed in standard form as1.5×10^8 km.

Example

Express the following numbers in standard form.

(a) 24 (b) 8693 (c) 0.0873 (d) 0.00506

Solutions

(a) $24 = 2.4 \times 10$

(b) $8693 = 8.693 \times 10^3$

(c) $0.0873 = 8.73 \times 10^{-2}$

(d) $0.00506 = 5.06 \times 10^{-3}$

Exercise 5.2

1. Express the following numbers in standard form.

   (a) 57600 (b) 0.0195

   (c) 300000 (d) 0.000000718

   (e) 0.00000198 (f) 998452

   (g) 1596.9 (h) 290.64

2. The following numbers are in standard form. Write them in their normal form.

   (a) 2.35×10^2 (b) 1.15×10^{-2}

   (c) 9.9×10^5 (d) 2.354×10^8

   (e) 1.56×10^{-5} (f) 1.0601×10^{-4}

3. The size of a molecule is estimated to be 0.0000001 mm. Express the size of the molecule in standard form.
4. A bacterium divides itself 120,000,000 times in 1 minute. Express the total number of bacteria reproduced within 1 minute in standard form.

5. The population of Uganda is expected to be 54.3 million in ten years' time. Express this population in standard form.

Multiplying decimals by 10, 100, 1000 …

When a number is multiplied by 10, tens become hundreds, units become tens, tenths become units, hundredths become tenths and so on. For example, $3.64 \times 10 = 36.4$.
In general, when a number is multiplied by 10, 100 or 1000, the digits move 1, 2, 3 places respectively to the left. Notice that the number of places moved is equal to the number of zeros in the multiplier.

Dividing decimals by 10, 100, 1000 …

When a number is divided by 10, hundreds become tens, tens become units, units become tenths, tenths become hundredths and so on. For example:

$120 \div 10 = 12$

$6.19 \div 10 = 0.619$

The digits move one place to the right and their values become one-tenths of their original values. Similarly, when you divide a number by 100, 1000, … the digits move 2, 3, … places to the right. In other words when we divide a decimal number by10, 100, 1000, … the decimal point moves 1, 2, 3 .. places respectively to the left. For example:

$5.83 \div 10 = 0.583$
$5.83 \div 100 = 0.0583$
$5.83 \div 1000 = 0.00583$

Notice that the number of places moved by the decimal point to the left is equal to the number of zeros in the divisor.

Exercise 5.3

Evaluate:

1. 0.158×100
2. $2.1 \times 10\,000$
3. 0.0091×10
4. $31.24 \times 1\,000$
5. $0.000027 \times 100\,000$
6. 1.4008×100
7. $4.7 \div 10$
8. $3.16 \div 1\,000$
9. $24.6 \div 100$
10. $0.097 \div 1\,000$
11. $0.2 \div 10\,000$
12. $10.15 \div 100\,000$

Indices

An expression like $3 \times 3 \times 3 \times 3$ can be written as 3^4. This is read as *three raised to the power of four*. The power is also called an **index** (plural **indices**) and the number 3 is the **base**.

Note: *An index is the number of times the base is multiplied by itself.*

Expressions can be written in index form as follows:

$2 \times 2 \times 2 \times 2 \times 2 = 2^5$
$5 \times 5 \times 5 \times 5 = 5^4$
$a \times a \times a \times a \times a \times a \times a = a^7$

The laws of indices

When numbers with the same base are multiplied or divided, their indices are operated according to certain rules.

Multiplication

$a^2 \times a^3 = (a \times a) \times (a \times a \times a)$
$= a \times a \times a \times a \times a$
$= a^5$.

Also, $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a)$
$= a \times a \times a \times a \times a \times a \times a$
$= a^7$

Note: *An index of a product is the sum of the given indices.*

Thus, $a^2 \times a^3 = a^{2+3} = a^5$
$a^3 \times a^4 = a^{3+4} = a^7$

In general, $a^m \times a^n = a^{m+n}$

For example, $4^2 \times 4^3 = 4^{2+3} = 4^5$

Division

$$a^5 \div a^3 = \frac{a \times a \times \cancel{a} \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a} \times \cancel{a}}$$

$$= a^2$$

Also, $$a^7 \times a^2 = \frac{a \times a \times a \times a \times a \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}}$$

$$= a^5.$$

Note: *The index of a quotient is the index of the divisor subtracted from the index of the divident*. Thus, $a^5 \div a^3 = a^{5-3} = a^2$

Also, $a^7 \times a^2 = a^{7-2} = a^5$

In general, $a^m \div a^n = a^{m-n}$

Powers

Consider the number $(a^2)^3$

We know that: $(a^2)^3 = a^2 \times a^2 \times a^2$

$= a^{2+2+2}$ or $a^{2\times3}$

$= a^6$

Also, $(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3$ or $a^{3\times4} = a^{12}$

Note that the letter *a* has been used in the above example to represent any number.

The three basic laws of indices are:

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

Exercise 5.4

Simplify, giving the results in index form.

1. $3^4 \times 3$
2. $4^6 \times 4^3$
3. $2^5 \times 2^3 \times 2^7$
4. $a^2 \times a^6$
5. $a^{12} \times a \times a^2$
6. $y^6 \times y^5 \times y^2$
7. $5^8 \div 5^4$
8. $2^5 \div 2^4$
9. $3^3 \div 3$
10. $7^9 \div 7^4$
11. $(2^6 \times 2^5) \div 2^4$
12. $2^6 \div (2^5 \times 2^4)$
13. $(5^4)^3$
14. $(c^6)^2$
15. $(3^4)^2$
16. $(5^3)^4$
17. $(u^{10})^3$
18. $(7^1)^9$
19. $(4)^6$
20. $(6^2)^{12}$
21. $u^{10} \div u^7 \times u$
22. $2^a \times 2^b \times 2$
23. $(e^5)^a$
24. $(3^n)^2$

Zero index

Consider the expression, $a^2 \div a^2$.

We know that $a^2 \div a^2 = \frac{\not{a} \times \not{a}}{\not{a} \times \not{a}} = 1$

And $a^2 \div a^2 = a^{2-2} = a^0$

Therefore, $a^0 = 1$.

In general, any number or expression with a power (or index) zero is equal to 1. For example, $3^0 = 1$; $150^0 = 1$ and $(a \times b)^0 = 1$.

Positive indices

Consider a product of numbers with an index. For example $(ab)^2$.

$$\begin{aligned} ab^2 &= ab \times ab \\ &= a \times b \times a \times b \\ &= a \times a \times b \times b \\ &= a^2 \times b^2 \\ &= a^2b^2 \end{aligned}$$

Similarly, $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$.

In general, $(ab)^n = a^nb^n$ and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example:

Simplify: (a) $(2a)^3$ (b) $(3y^2)^4$

(c) $2a^5 \times 3a^4$

Solutions

(a) $(2a)^3 = 2^3a^3 = 8a^3$

(b) $(3y^2)^4 = 3^4 \times (y^2)^4$

$= 3^4y^8$

$= 81y^8$

(c) $2a^5 \times 3a^4 = 2 \times 3 \times a^5 \times a^4$

$= 6a^9$

Example

Simplify: $\frac{24a^6 \times 5a^3}{8a^7}$

Solution

$\frac{{}^{3}\not{24}a^6 \times 5a^3}{{}_{1}\not{8}a^7}$ simplifying coefficient by cancellation

$= 3a^6 \times 5a^3 \div a^7 = 15a^{6+3-7}$

$= 15a^2$

Exercise 5.5

Simplify, giving the results in index form.

1. $2a^3 \times 5a^4$
2. $a^5 \times 3a^7$
3. $2m^2 \times n^2 \times m^3$
4. $\frac{1}{4}a^2 \times 12a^3 \times 6a^6$
5. $3a^4 \times 2b^2 \times 5a^3 \times 4b^5$
6. $6p^6 \times 5q \times q^2 \times p$
7. $7y^0 \times 3z^4 \times 4z^0$
8. $4a^5 \div 2a^3$
9. $18a^7 \div 6a^5$
10. $\dfrac{10y^4 \times 9y^3}{3y^2}$
11. $\dfrac{15t^5 \times 6t^5}{5t^7}$
12. $\dfrac{4t^7 \times 3t^2 \times t^3}{t^2 \times 6t^5}$
13. $(2y^3)^5$
14. $(5z^2)^3$
15. $\left(\frac{1}{2}y^3\right)^5$
16. $(ab^2)^3$
17. $\left(\dfrac{a}{b^3}\right)^4$
18. $(2pq^4)^3$
19. $\left(\frac{4}{x}\right)^2$
20. $\left(\frac{1}{y}\right)^0$

Negative indices

Consider the expression: $a^3 \div a^5 = a^{3-5} = a^{-2}$.

We know that $a^3 \div a^5 = \dfrac{a \times a \times a}{a \times a \times a \times a \times a} = \dfrac{1}{a^2}$

Thus, $\dfrac{1}{a^2} = a^{-2}$

In general, $a^{-n} = \dfrac{1}{a^n}$.

Note: The laws of indices apply to both positive and negative indices.

Example

Simplify:

(a) $2^{-3} \times 2^{-2}$

(b) $3^{-2} \times 3^{-5}$

(c) $a^{-4} \div a^{-2}$

(d) $a^6 \div a^{-3}$

(e) $(a^{-2})^3$

Solutions

(a) $2^{-3} \times 2^{-2} = 2^{-3+(-2)} = 2^{-3-2} = 2^{-5}$

(b) $3^{-2} \times 3^{-5} = 3^{-2+(-5)} = 3^{-2-5} = 3^{-7}$

(c) $a^{-4} \div a^{-2} = a^{-4-(-2)} = a^{-4+2} = a^{-2}$

(d) $a^6 \div a^{-3} = a^{6-(-3)} = a^{6+3} = a^9$

(e) $(a^{-2})^3 = a^{-2\times 3} = a^{-6}$

Example

Express the following with positive indices:

(a) 5^{-3} (b). a^{-4}

(c) $\left(\frac{a}{b}\right)^{-3}$ (d) $2a^{-5}$

(e) $(3a)^{-2}$ (f) $\frac{1}{u^{-2}}$

Solutions

(a) $5^{-3} = \frac{1}{5^3}$ (b) $a^{-4} = \frac{1}{a^4}$

(c) $\left(\frac{a}{b}\right)^{-3} = 1 \div \left(\frac{a}{b}\right)^3 = 1 \times \left(\frac{b}{a}\right)^3 = \frac{b^3}{a^3}$

(d) $2a^{-5} = 2 \times \frac{1}{a^5} = \frac{2}{a^5}$

(e) $(3a)^{-2} = \frac{1}{(3a)^2} = \frac{1}{9a^2}$

(f) $\frac{1}{u^{-2}} = 1 \div u^{-2} = 1 \div \frac{1}{u^2} = 1 \times \frac{u^2}{1} = u^2$

Note: In (d), -5 is the index for a only while in (e), -2 is the index for both 3 and a.

Exercise 5.6

1. Simplify, leaving your answers in index form.

(a) $7^{-2} \times 7^{-2}$ (b) $3^{-5} \times 3^{-4}$

(c) $a^3 \times a^{-5}$ (d) $a^{-7} \times a^4$

(e) $2^{-3} \div 2^4$ (f) $5^{-2} \div 5^{-6}$

(g) $a^{-11} \div a^{-8}$ (h) $a^3b^2 \div a^4b^5$

2. Express the following as positive indices:

(a) 2^{-4} (b) $2^{-3}a^{-5}$

(c) $3a^{-4}$ (d) $\frac{1}{a^{-2}}$

(e) $(a^{-2})^{-3}$ (f) $(a^{-1})^2$

(g) $(a^2b^{-3})^{-1}$

Fractional indices

The laws of indices also apply to fractions. We have seen that

$a^m \times a^n = a^{m+n}$.

Also consider $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

We have multiplied $a^{\frac{1}{2}}$ by itself to obtain a or $a^{\frac{1}{2}}$ has been squared to get a.

Thus, $\left(a^{\frac{1}{2}}\right)^2 = a^{\frac{1}{2}\times 2}$ *the third law of indices*

$= a$

Remember that the square of 4 is 16. Hence, 4 is the square root of 16, written as $\sqrt{16}$.

In general, $\sqrt{a} \times \sqrt{a} = a$.

The number $\sqrt{a}$ is also written as $a^{\frac{1}{2}}$.

Similarly, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$

Thus, $\left(a^{\frac{1}{3}}\right)^3 = a^{\frac{1}{3}\times 3} = a$.

This means that $a^{\frac{1}{3}}$ is the cube root of a and is also written as $\sqrt[3]{a}$.

In general, the n^{th} root of a is written as $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$. From the laws of indices we have, $(a^m)^n = (a^n)^m = a^{mn}$.

You should be able to show that $(a^m)^n = (a^n)^m$.

Therefore, $\left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = a^{\frac{m}{n}}$.

Example

Simplify:

(a) $36^{\frac{1}{2}}$

(b) $64^{\frac{1}{3}}$

(c) $125^{\frac{1}{3}}$

(d) $\left(\frac{8}{27}\right)^{\frac{1}{3}}$

Solutions

(a) $36^{\frac{1}{2}} = \left(6^2\right)^{\frac{1}{2}} = 6^{2\times\frac{1}{2}} = 6^1 = 6$

(b) $64^{\frac{1}{3}} = \left(4^3\right)^{\frac{1}{3}} = 4^{3\times\frac{1}{3}} = 4^1 = 4$

(c) $125^{\frac{1}{3}} = \left(5^3\right)^{\frac{1}{3}} = 5^{3\times\frac{1}{3}} = 5$

(d) $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \left(\frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}}\right) = \frac{2}{3}$

Remember that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example

Simplify:

(a) $8^{\frac{2}{3}}$

(b) $32^{\frac{2}{5}}$

(c) $\left(\frac{81}{64}\right)^{\frac{-3}{2}}$

Solutions

(a) $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$

Alternatively, $8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}} = 64^{\frac{1}{3}} = 4$

(b) $32^{\frac{2}{5}} = \left(32^{\frac{1}{5}}\right)^2 = 2^2 = 4$

(c) $\left(\frac{81}{64}\right)^{\frac{-3}{2}} = \left(\frac{64}{81}\right)^{\frac{3}{2}}$, since $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

$$= \left(\left(\frac{64}{81}\right)^{\frac{1}{2}}\right)^3 = \left(\frac{8}{9}\right)^3 = \frac{512}{729}$$

Example

Simplify:

(a) $y^{\frac{1}{2}} \times y^{\frac{-1}{3}}$

(b) $5^{\frac{-1}{2}} \times 20^{\frac{1}{2}}$

(c) $8a^{\frac{1}{2}} \div 12a^{\frac{5}{2}}$

Solutions:

(a) $y^{\frac{1}{2}} \times y^{\frac{-1}{3}} = y^{\frac{1}{2}+\left(\frac{-1}{3}\right)}$

$$= y^{\frac{1}{2}-\frac{1}{3}} = y^{\frac{1}{6}}$$

(b) $5^{\frac{-1}{2}} \times 20^{\frac{1}{2}} = \left(\frac{1}{5}\right)^{\frac{1}{2}} \times 20^{\frac{1}{2}}$

$$= \left(\frac{1}{5} \times 20\right)^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}} = 2$$

(c) $8a^{\frac{1}{2}} \div 12a^{\frac{5}{2}} = \frac{8a^{\frac{1}{2}}}{12a^{\frac{5}{2}}} = \frac{8}{12}.a^{\frac{1}{2}-\frac{5}{2}}$

$$= \frac{2}{3}.a^{-2}$$

$$= \frac{2}{3a^2}$$

Example

Solve for x if: (a) $3^x = 243$ (b) $2^{x-1} = \frac{32}{\sqrt{2}}$ (c) $x^{\frac{-3}{4}} = 27$

Solution:

(a) First express 243 in index notation with base 3. That is, $243 = 3^5$. Thus, $3^x = 3^5$. Since the bases are the same, their indices are equal. Therefore, x = 5.

(b) $2^{x-1} = \frac{32}{\sqrt{2}} = 32 \div 2^{\frac{1}{2}} = 2^5 \div 2^{\frac{1}{2}}$

$= 2^{5-\frac{1}{2}}$

Therefore, $2^{x-1} = 2^{4\frac{1}{2}}$

Equating the indices we get,

$x - 1 = 4\frac{1}{2}$

$x = 5\frac{1}{2}$

(c) $x^{\frac{-3}{4}} = 27 \Leftrightarrow x^{\frac{-3}{4}} = 3^3$

We can make the power of x be 1. This can be done by multiplying both indices by $\frac{-4}{3}$, which is the reciprocal of $\frac{-3}{4}$.

Thus, $\left(x^{\frac{-3}{4}}\right)^{\frac{-4}{3}} = (3^3)^{\frac{-4}{3}}$

$x^{\frac{-3}{4}\times\frac{-4}{3}} = 3^{3\times\frac{-4}{3}}$

$x = 3^{-4}$

$= \frac{1}{81}$

Summary of the laws of indices.

1. $a^m \times a^n = a^{m+n}$

2. $a^m \div a^n = a^{m-n}$

3. $(a^n)^m = (a^m)^n = a^{mn}$

4. $a^{-n} = \frac{1}{a^n}$

5. $a^0 = 1$

6. $(ab)^n = a^n b^n$

7. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Exercise 5.7

1. Simplify:

(a) $49^{\frac{3}{2}}$

(b) $16^{\frac{1}{4}}$

(c) $5^{\frac{1}{3}} \times 5^{\frac{1}{4}}$

(d) $32^{\frac{2}{5}} \times 243^{\frac{1}{5}}$

(e) $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 6^{\frac{1}{2}}$

(f) $a^{\frac{4}{5}} \div a$

(g) $a^{\frac{1}{6}} \div a^{\frac{1}{3}}$

(h) $(a^2 \times a^6)^{\frac{1}{4}}$

(i) $\left(a^{5} \div a^{-1}\right)^{\frac{1}{2}}$ (j) $5y^{\frac{5}{2}} \times 2y^{\frac{1}{2}}$

(k) $\left(16a^{-4}\right)^{\frac{1}{2}}$ (l) $(63a)^{\frac{1}{2}} \div \left(7a^{2}\right)^{\frac{1}{2}}$

(m) $1 \div 3a^{\frac{-1}{3}}$ (n) $\left(\frac{125}{343}\right)^{\frac{-2}{3}}$

(o) $\left(\frac{1}{729}\right)^{\frac{-1}{3}}$ (p) $\left(3^{4}\right)^{\frac{-3}{2}}$

(q) $\left(6a^{4}\right)^{\frac{1}{2}} \times \left(\frac{a}{6}\right)^{\frac{-1}{2}}$ (r) $5\left(a^{3}\right)^{\frac{-2}{3}} \times 4a^{\frac{1}{2}}$

(s) $\dfrac{32^{\frac{2}{5}} \times 27^{\frac{1}{3}}}{18}$

2. Solve for a in:

(a) $2^{a} = \frac{1}{64}$ (b) $3^{a} = 27^{\frac{5}{3}}$

(c) $3 \times 2^{a+5} = 768$ (d) $25^{4a} = 125$

(e) $\left(\frac{1}{4}\right)^{a-3} = 1024$ (f) $a^{\frac{-1}{3}} = 4$

(g) $a^{\frac{2}{3}} = 16$ (h) $2a^{\frac{1}{2}} = 16$

3. If a = 8 and b = 36 calculate the values of:

(a) $a^{\frac{1}{3}} b^{\frac{-3}{2}}$ (b) $\left(a^{\frac{-2}{3}} + b^{\frac{1}{2}}\right)^{\frac{1}{2}}$

4. Evaluate:

(a) $32^{1\frac{1}{5}}$ (b) 5^{-5}

(c) $81^{\frac{1}{4}}$ (d) $\left(\frac{16}{250}\right)^{\frac{1}{3}}$

(e) $(4abc)^{0}$ (f) $(-1)^{-3}$

(g) $\left(1\frac{1}{2}\right)^{-4}$ (h) $\frac{26x^{7}}{4x^{5}}$

(i) $\left(25x^{6}\right)^{\frac{3}{2}}$ (j) $\sqrt{36a^{-8}}$

(k) $c \times c^{\frac{1}{2}}$ (l) $r^{\frac{3}{4}} \div r^{\frac{1}{4}}$

(m) $2a^{4} \times 5^{a-3}$ (n) $m \div m^{-2}$

(o) $9^{2n} \div 3^{n}$ (p) $9^{n-1} \times 27^{n+3}$

(q) $(a^{2} + b^{3})^{0}$

5. Find a and b if $2^{a} \times 3^{b} = 12$.

6. Find a, if $6^{a} = \frac{1}{216}$.

7. Find n, if $4^{2n-1} = 1024$

Chapter 6.

SURDS

A number which cannot be expressed as an integer, a terminating decimal or a recurring decimal is called an *irrational number*.

$\sqrt{2}$ cannot be expressed as a fraction, i.e. it is neither a terminating nor a recurring decimal.

$\sqrt{2} = 1.414213562...$

Therefore, $\sqrt{2}$ is an irrational number.

Other roots such as $\sqrt{3}$, $\sqrt{11}$, $\sqrt[3]{4}$ . . . are examples of irrational numbers.

Now consider the following cases:

(a) $\sqrt{4} = 2$; $\sqrt[3]{27} = 3$; $\sqrt[4]{16} = 2$; $\sqrt{0.09} = 0.3 = \frac{3}{10}$.

(b) $\sqrt{2} = 1.414213562. . .$; $\sqrt{5} = 2.236067 . . .$;

We observe that:

In (a), the numbers under the root sign are positive rational numbers and the roots of these numbers are either integers or can be expressed as a ratio of two integers.

In (b), the numbers under the root sign are positive rational numbers but the roots of these numbers are irrational. Such numbers are called **surds**.

A surd is an irrational root of a rational number.

Thus $\sqrt{16} = 4$ is not a surd but $\sqrt[3]{16}$ is a surd.

$\sqrt{8}$ is a surd but $\sqrt[3]{8}$ is not a surd.

Simplification of surds

(a) Reduced form of a surd

Consider the following example:

$\sqrt{72} = \sqrt{36 \times 2} = \sqrt{36} \times \sqrt{2} = 6 \times \sqrt{2} = 6\sqrt{2}$.

Example 1

Reduce to the simplest form of a surd:

(i) $\sqrt{8}$ (ii) $\sqrt{45}$

Solutions

(i) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}.$

(ii) $\sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}.$

(b) Surds of the same order can be multiplied or divided.

$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$

$\sqrt{a} \times \sqrt{a} = \sqrt{a \times a} = \sqrt{a^2} = a$

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\left(\frac{a}{b}\right)}$

Example 2

Simplify: (i) $\sqrt{8}\times\sqrt{2}$, (ii) $\frac{\sqrt{12}}{\sqrt{27}}$

Solutions

(i) $\sqrt{8}\times\sqrt{2}=\sqrt{16}=4.$

(ii) $\frac{\sqrt{12}}{\sqrt{27}} = \sqrt{\left(\frac{12}{27}\right)}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

Addition and subtraction of surds

Surds of the same order with the same number under the root sign can be added or subtracted.

Example 3.

Simplify: (i) $\sqrt{5}+\sqrt{20}$ (ii) $\sqrt{54}+\sqrt{150}-\sqrt{24}$

Solutions

(i) $\sqrt{5}+\sqrt{20} = \sqrt{5}+\sqrt{4\times5}=\sqrt{5}+2\sqrt{5}=\mathbf{3\sqrt{5}}$.

(ii) $\sqrt{54}+\sqrt{150}-\sqrt{24} = \sqrt{9\times6}+\sqrt{25\times6}-\sqrt{4\times6}$

$= 3\sqrt{6}+5\sqrt{6}-2\sqrt{6}$

$= \mathbf{6\sqrt{6}}$.

In simplification of surds the following rules are used:

(i) $a\sqrt{c}+b\sqrt{c}=(a+b)\sqrt{c}$.

(ii) $a\sqrt{c}-b\sqrt{c}=(a-b)\sqrt{c}$

(iii) $\sqrt{a}\times\sqrt{a}=a$

(iv) $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$

(v) $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

Note that $\sqrt{a}\pm\sqrt{b}$ is not the same as $\sqrt{a\pm b}$.
For example, $\sqrt{9+16}=\sqrt{25}=5$
But $\sqrt{9+16}\neq\sqrt{9}+\sqrt{16}$ as $\sqrt{9}+\sqrt{16}=3+4=7$.

Exercise 6.1

Express the following irrational numbers so that the integer under the square root sign is as small as possible, e.g. $\sqrt{12}=\sqrt{4\times3}=2\sqrt{3}$.

1. $\sqrt{24}$
2. $\sqrt{180}$
3. $\sqrt{32}$
4. $\sqrt{108}$

5. $\sqrt{800}$ 6. $\sqrt{720}$

7. $\sqrt{5000}$ 8. $\sqrt{74a^8}$

9. $\sqrt{50}$ 10. $\sqrt{147}$

11. $\sqrt{135}$ 12. $\sqrt{150}$

13. $\sqrt{24a^3b^2}$ 14. $\sqrt{18x^4y^3z}$

15. $\sqrt{1000p^2q}$

Express the following as complete square roots:
e.g. $2\sqrt{3}=\sqrt{4\times3}=\sqrt{12}$.

16. $5\sqrt{3}$ 17. $\frac{1}{3}\sqrt{54}$

18. $4\sqrt{12}$ 19. $2a\sqrt{b}$

20. $4\sqrt{5}$ 21. $10\sqrt{7}$

22. $\frac{2}{3}\sqrt{135}$ 23. $3x^2\sqrt{5}$

Simplify the following:

24. $\sqrt{14}\times\sqrt{7}$ 25. $\sqrt{32}\times\sqrt{24}$

26. $\sqrt{5}\times2\sqrt{5}$ 27. $\sqrt{3}\times\sqrt{15}$

28. $2\sqrt{8}\times3\sqrt{8}$ 29. $2\sqrt{27}\times3\sqrt{6}$

30. $2\sqrt{20}\times3\sqrt{7}$ 31. $\sqrt{10}\times2\sqrt{2}\times\sqrt{5}$

32. $\sqrt{10}\times\sqrt{30}\div\sqrt{12}$

Rationalization of the denominator

A number such as $\frac{1}{\sqrt{2}}$ can easily be evaluated by first expressing it with a rational denominator. The process of converting an irrational denominator to a rational one is known as **rationalization**.

Example 4

Evaluate: (i) $\frac{1}{\sqrt{2}}$ (ii) $\frac{12}{\sqrt{18}}$, given that $\sqrt{2}\approx1.4142$.

Solutions:

(i) $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}\approx\frac{1.4142}{2}=\mathbf{0.7071}$

(ii) $\frac{12}{\sqrt{18}} = \frac{12}{\sqrt{9\times2}} = \frac{12}{3\times\sqrt{2}}=\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2}$.

$= 2\sqrt{2} = 2 \times 1.4142 =$ **2.8284.**

Example 5

Simplify $\frac{\sqrt{5}}{2\sqrt{3}}$ by rationalizing the denominator.

Solution

$\frac{\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{5}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{15}}{2\times 3} = \frac{\sqrt{15}}{6}$.

Example 6

Rationalise the denominator of $\frac{1}{\sqrt{5}+2}$.

Solution:

Note: When the denominator is of the form $a \pm b\sqrt{c}$, where a, b and c are integers, we multiply both the numerator and denominator by the **conjugate surd** of the denominator.

$a - b\sqrt{c}$ **is the conjugate surd of** $a + b\sqrt{c}$ and vice versa.

Hence the conjugate surd of $\sqrt{5}+2$ is $\sqrt{5}-2$.

Therefore, $\frac{1}{\sqrt{5}+2} = \frac{1}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2} = \frac{\sqrt{5}-2}{\sqrt{5}(\sqrt{5}-2)+2(\sqrt{5}-2)} = \frac{\sqrt{5}-2}{5-4}$

$= \sqrt{5}-2$

Exercise 6.2

Express the following with rational denominators:

1. $\frac{2}{\sqrt{3}}$
2. $\frac{2}{\sqrt{6}}$
3. $\frac{1}{\sqrt{5}}$
4. $\frac{\sqrt{2}}{\sqrt{3}}$
5. $\frac{10}{\sqrt{2}}$
6. $\frac{3}{2\sqrt{5}}$
7. $\frac{3}{\sqrt{10}}$
8. $\frac{8}{\sqrt{12}}$
9. $\frac{10}{\sqrt{20}}$
10. $\frac{a}{\sqrt{a}}$

Evaluate the following, without using calculator, correct to 4 SF, given that $\sqrt{2}$ = 1.4141 and $\sqrt{3}$ = 1.7321

11. $\frac{5}{\sqrt{2}}$
12. $\frac{4}{\sqrt{3}}$
13. $\frac{2\sqrt{6}}{\sqrt{3}}$
14. $\frac{5}{2\sqrt{2}}$

Simplify:

15. $3\sqrt{12}-2\sqrt{3}$

16. $\sqrt{12}+3\sqrt{75}$

17. $\sqrt{12}-2\sqrt{3}$

18. $\sqrt{18}-\sqrt{200}-2\sqrt{72}$

19. $\sqrt{50}-\sqrt{98}-\sqrt{8}$

20. $\sqrt{3}(3+\sqrt{3})$

21. $\sqrt{3}(3-\sqrt{27})$

22. $\sqrt{45}+3\sqrt{20}-\sqrt{80}$

23. $2\sqrt{45}+\sqrt{20}-3\sqrt{80}$

24. $5\sqrt{6}+\sqrt{294}-\sqrt{24}$

25. $\sqrt{175}-2\sqrt{28}+\sqrt{63}$

26. $\sqrt{a}(\sqrt{a}-\sqrt{b})$

27. $(7-3\sqrt{2})(7+3\sqrt{2})$

28. $(\sqrt{2}+2\sqrt{3})(\sqrt{2}-2\sqrt{3})$

Rationalise the denominators (Q.29 - 39)

29. $\frac{1}{\sqrt{5}-1}$

30. $\frac{3}{\sqrt{7}-\sqrt{2}}$

31. $\frac{4}{6-\sqrt{5}}$

32. $\frac{3}{2\sqrt{3}-\sqrt{2}}$

33. $\frac{\sqrt{2}}{3-\sqrt{2}}$

34. $\frac{\sqrt{3}}{\sqrt{3}-1}$

35. $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$

36. $\frac{2\pm\sqrt{11}}{\sqrt{11}+4}$

37. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

38. $\frac{6\sqrt{2}}{\sqrt{10}-\sqrt{7}}$

39. $\frac{\sqrt{33}-\sqrt{23}}{\sqrt{33}+\sqrt{23}}$

If x = $\sqrt{3}+2$ and y = $\sqrt{3}-2$, find the value of: (Q.40 - 42)

40. xy

41. $x^2 + y^2$

42. $x^2 - y^2$

43. Evaluate: $x^2 + 2x$, if x = $\sqrt{5}$ and $\sqrt{5}$ = 2.2361.

44. Simplify: $x^2 + (\sqrt{3})x$ if x = 2 - $\sqrt{3}$.

Simplify the following by rationalizing the denominators Q.45 - 46:

45. $\frac{1}{\sqrt{5}\pm 2}+\frac{1}{\sqrt{5}+2}$

46. $\frac{1}{\sqrt{7}\pm\sqrt{3}}-\frac{1}{\sqrt{7}+\sqrt{3}}$

Chapter 7.

LOGARITHMS

Powers of 10 and logarithms

Numbers like 10, 100, 1 000, 10 000 and so on, can easily be expressed as powers of 10.
Thus: $10 = 10^1$, $100 = 10^2$, $1\ 000 = 10^3$ and $10\ 000 = 10^4$.

The power of 10 for a given number is its logarithm to base 10. Thus, the logarithm of 1 000 to base 10 is 3, written as
$\log_{10} 1000 = 3$ or simply log 1 000 = 3.

The logarithm of a number is the power to which 10 is raised to give that number.
Therefore, $1\ 000 = 10^3 = 10^{\text{logarithm}}$.
In general, if 10 is raised to power *a* to give x then $x = 10^a$. This means that the logarithm of x to base 10 is *a*, written as $\log_{10} x = a$ or simply log x = a.

The logarithms of some numbers are integers while the logarithms of other numbers are not integers. However, there are tables which enable us to find the logarithms of numbers.

Logarithms of numbers between 1 and 10.

These can be obtained from the table of logarithms.

Example 1

Find the logarithm to base 10 of 1.86

Solution

From the table of logarithms shown below, find the logarithm of 1.86 as follows:
Look for the number 1.8 in the column headed x. Move along this row up to where it meets with the main column headed 6. Read the number at the intersection of the row and the column. The number is 0.2695.
Therefore, log 1.86 = 0.2695 or $1.86 = 10^{0.2695}$

Example 2

Find $\log_{10} 2.356$

Solution

The number has four significant figures. Look for the number 2.4 in the column headed x. Move along this row up to where it meets with the main column headed 5. Read the number at the intersection. The number is 0.3711. Move further along the same row up to the differences column headed 6. The number at this intersection is 11. Write this number as 0.0011. Add 0.0011 to 0.3711 to get 0.3722.
Therefore, $\log_{10} 2.356 = 0.3722$ or $10^{0.3722} = 2.456$.

Table 7.1: Part of a table of logarithms to base 10

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Differences (ADD)								
1.0	0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1.4	0.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2.0	0.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2.3	0.3619	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17

Exercise 7.1

1. Express the following numbers as powers of 10:

(a) 100 (b) 100 000

(c) 1 (d) 1.53

(e) 7.002 (f) 6.23

(g) 8.1 (h) 1.01

2. Find the logarithms of the following numbers:

(a) 9.991 (b) 1.025

(c) 3.108 (d) 4.444

(e) 7 (f) 6.002

Logarithms of numbers greater than 10

Numbers that are greater than 10 are first expressed in standard form in order to get their logarithms. That is, A × 10^n, where $1 \leq A \leq 10$ and then the laws of indices are applied.

Example 3

Find the logarithms of:

(a) 27.4 (b) 382

(c) 18 347

Solutions:

(a) $27.4 = 2.74 \times 10^1$

From tables, log 2.74 = 0.4378

Therefore, $27.4 = 10^{0.4378} \times 10^1 = 10^{0.4378+1} = 10^{1.4378}$

Therefore, log 27.4 = 1.4378.

(b) A number with more than 4 significant figures should first be rounded to 4 significant figures.
Thus, 18 347 = 18 350 (to 4 s.f.)
$18\ 350 = 1.835 \times 10^4$
Fro tables, log 1.835 = 0.2637
So $18\ 350 = 10^{0.2637} \times 10^4 = 10^{4.2637}$
Therefore, log 18 350 = 4.2637.

A logarithm has two parts:

(a) the integer part before the decimal point, called the **characteristic**.
(b) the decimal part after the decimal point called the **mantissa**.

Note: *The characteristic of a logarithm is the same as the power of 10 when the number is written in standard form.*

Exercise 7.2

Find the logarithms of the following numbers.

1. 17	**2.** 34.6
3. 485	**4.** 3094
5. 1006	**6.** 597.2
7. 20.17	**8.** 800.4
9. 88.8	**10.** 543 000
11. 123 456	**12.** 49 836
13. 8	**14.** 2.8
15. 9.05	**16.** 3 014 952
17. 100.02	**18.** 111.11

Logarithms of numbers between 0 and 1

The logarithms of numbers between 0 and 1 can be found from tables of logarithms by first expressing them in standard form.

Example 4

Find the logarithms of:
(a) 0.723 (b) 0.0052

Solutions

(a) $0.723 = 7.23 \times 10^{-1}$
From tables, log 7.23 = 0.8591.
Thus, $0.723 = 10^{0.8591} \times 10^{-1}$ or $10^{-1+0.8591}$
$= 10^{0.8591 + (-1)} = 10^{\bar{1}.8591}$
Therefore, log 0.723 = $\bar{1}.8591$.

$\bar{1}.8591$ is read as *bar one point eight five nine one.*

Note: Only the characteristic is negative. The mantissa is always positive. Therefore, the negative sign is put above the characteristic.

(b) $0.0052 = 5.2 \times 10^{-3}$

From the tables, log 5.2 = 0.7160

Thus, $0.0052 = 10^{0.7160} \times 10^{-3}$

$= 10^{-3+0.7160}$

$= 10^{\bar{3}.7160}$

Therefore, log 0.0052 = $\bar{3}.7160$

Exercise 7.3

Find the logarithms of the following numbers:

1. 0.44
2. 0.9
3. 0.005
4. 0.0089
5. 0.000046
6. 0.001
7. 0.178
8. 0.0102
9. 42
10. 0.0000008
11. 0.002059
12. 5.7
13. 601
14. 1 000 000
15. 2.003

Antilogarithms

The logarithm of 2.4 is 0.3802. The number whose logarithm is 0.3802, is called the antilogarithm of 0.3802, which is 2.4.
Thus, antilogarithm 0.3802 = 2.4. Simply written as antilog 0.3802 = 2.4.
Tables of antilogarithms are also given. See Table 7.2. The tables are used in the same way as those of logarithms

Table 7.2: Part of a table of antilogarithms to base 10											Differences (ADD)								
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1.000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1.023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1.047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1.072	1074	1076	1079	1081	1084	1086	1086	1091	1094	0	0	1	1	1	1	2	2	2
.04	1.096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1.122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1.148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1.175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1.202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1.230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1.259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3

Example 5

Find the numbers whose logarithms are:

(a) 0.1068 (b) 2.0088

(c) $\bar{1}.0769$

Solution

(a) Look for the number 0.10 on the column headed x. Move along this row until where it meets with the main column headed 6. The number at the intersection is 1.276. Continue along this row up to the differences column head 8. The number at this intersection is 2. Write this number as 0.002. Add 0.002 to 1.276 to get 1.278.
Therefore, antilog 0.1068 = 1.278.

Note: *The decimal points are only indicated in the column headed 0. These decimal points apply in the whole table.*

(b) Similarly, from the tables (see Table 7.2):
Antilog 0.008 = 1.019
The difference for 8 = 0.002
1.019 + 0.002 = 1.021
Therefore, antilog 2.0088 = 1.021×10^2.
(in 2.0088, *the characteristic* is 2)

(c) From the tables, antilog 0.076 = 1.191.
The difference for 9 = 0.002
1.191 + 0.002 = 1.193

Therefore, antilog $\bar{1}.0769 = 1.193 \times 10^{-1}$.
(in $\bar{1}.0769$, *the characteristic is* $\bar{1}$)

Exercise 7.4

Use tables to find the numbers whose logarithms are:

1. 0.6841 **2.** 0.0431

3. 1.9820 **4.** 3.1172

5. $\bar{2}.2003$ **6.** $\bar{4}.3142$

7. $\bar{1}.6990$ **8.** $\bar{3}.6246$

9. 5.3010 **10.** $\bar{6}.0000$

11. $\bar{1}.5752$ **12.** 0.9909

Application of logarithms

The main use of logarithms is in the multiplication and division of numbers. They are also used in finding the roots of numbers.

Multiplication of logarithms

Logarithms can be used to work out the products of numbers. We have already seen that logarithms are powers or indices. The rules of indices are therefore followed when working out multiplication using logarithms.

Example 6

Evaluate 4.5 × 6.27 using logarithms.

Solution

From tables, log 4.5 = 0.6532 and log 6.27 = 0.7973.

$4.5 \times 6.27 = 10^{0.6532} \times 10^{0.7973}$
$= 10^{0.6532+0.7973}$
$= 10^{1.4505}$

Therefore, log (4.5 × 6.27) = 1.4505.

From the table of antilogarithms, the number whose logarithm is 1.4505 or, antilog 1.4505 = 2.819×10^1 = 28.19.

Note: *The logarithm of 4.5 × 6.27 was obtained by adding log 4.5 to log 6.27.*

Log (4.5 × 6.27) = log 4.5 + log 6.27
= 0.6532 + 0.7973
= 1.4505

Generally, **log(ab) = log a + log b**

Note:The logarithm of a product of numbers is the sum of the logarithms of the numbers.

Example 6 can also be done in tabular form as follows:

No.	logarithm
4.5	0.6532 +
6.27	0.7973
28.19	1.4505

Antilog 1.4505 = 2.819×10^1
= 28.19

Example 7

Use logarithms to evaluate 356 × 43.6.

Solution

No.	Standard form	logarithm
356	3.56×10^2	2.5514 +
43.6	4.36×10^1	1.6395
15520	1.552×10^4	4.1909

Antilog 4.1909 is 1.552×10^4. Therefore, 456 × 43.6 = 15520

Example 8

Use logarithms to evaluate 0.0417 × 0.00928.

Solution

No.	Standard form	logarithm
0.0417	4.17×10^{-2}	$\bar{2}.6201$ +
0.00928	9.28×10^{-3}	$\bar{3}.9675$
0.0003869	3.869×10^{-4}	$\bar{4}.5876$

Antilog $\bar{4}.5876$ is 3.869×10^{-4}

$0.0417 \times 0.00928 = 3.869 \times 10^{-4} = 0.0003869$.

Exercise 7.5

Evaluate the following using logarithms:

1. 2.8×3.61
2. $13.6 \times 5.2 \times 6.2$
3. 30.7×10.71
4. 0.83×0.052
5. $1.346 \times 1.117 \times 2$
6. $4.2 \times 13.5 \times 9.1$
7. $0.0108 \times 1.76 \times 0.21$
8. 5.234×7.04
9. 183×356
10. 5400×1.36
11. 0.000812×0.049
12. 409×15.2
13. 0.0252×328
14. $14.7 \times 1.68 \times 44.7$

Division

As in multiplication, logarithms can be used to work out division of numbers.

Example 9

Evaluate $5.23 \div 3.14$

Solution

From the tables, log 5.23 = 0.7185 and log 3.14 = 0.4969.

Therefore, $5.23 \div 3.14 = 10^{0.7185} \div 10^{0.4949}$

$= 10^{0.7185 - 0.4969}$

$= 10^{0.2216}$

Therefore, log (5.23 ÷ 3.14) = 0.2216.

From the table of antilogarithms,

Antilog 0.2216 = 1.666.

Therefore, 5.23 ÷ 3.14 = 1.666.

Note that log (5.23 ÷ 3.14) = log 5.23 - log 3.14

= 0.7185 - 0.4969

= 0.2216

Generally, **log (a ÷ b) = log a - log b**. Or $\log\left(\frac{a}{b}\right) = \log a - \log b$.

Example 9 can also be done in a tabular form as follows:

No.	Logarithm
5.23	0.4969
3.14	0.7185 -
1.666	0.2216

Antilog of 0.2216 is 1.666. Therefore, 5.23 ÷ 3.14 = 1.666

Example 10

Use logarithms to evaluate 430 ÷ 38.

Solution

No.	Standard form	logarithm
430	4.3×10^2	2.6335 -
38	3.8×10^1	1.5798
11.32	1.132×10^1	1.0537

Antilog 1.0537 = 1.132×10^1. Therefore, 430 ÷ 38 = 11.32.

Example 11

Use logarithms to evaluate 0.0231 ÷ 0.000458.

Solution

No.	Standard form	logarithm
0.0231	2.31×10^{-2}	$\bar{2}.3636$ -
0.000458	4.58×10^{-4}	$\bar{4}.6609$
50.43	5.043×10^1	1.7027

Antilog 1.7027 = 5.043×10^1. Therefore, 0.0231 ÷ 0.000458 = 50.43

Exercise 7.6

Use logarithms to evaluate the expressions:

1. 3.4 ÷ 2.3

2. 8.33 ÷ 13.58

3. 1.03 ÷ 4.92

4. 832 ÷ 59.1

5. 14.3 ÷ 0.017

6. 0.314 ÷ 0.128

7. 0.0195 ÷ 0.00152

8. 0.0308 ÷ 0.07

Combined multiplication and division

Example 12

Use logarithms to evaluate:

(a) $\frac{4.2 \times 8.7}{5.6}$ (b) $\frac{743.1 \times 34.8}{15.6 \times 102.7}$ (c) $\frac{0.031 \times 0.00123}{0.215 \times 0.0802}$

Solutions

(a)

No.	logarithm
4.2 8.7	0.6232 0.9395 +
4.2 × 8.7 5.6	1.5627 0.7482 -
6.524	0.8145

Antilog 0.8145 = 6.524. Therefore, $\frac{4.2 \times 8.7}{5.6}$ = 6.524.

(b)

No.	Standard form	logarithm	
743.1 34.8	7.431×10^2 3.48×10^1	2.8710 1.5416	
743.1 × 34.8		4.4126	4.4126
15.6 102.7	1.56×10^1 1.027×10^2	1.1931 2.0116	-
15.6 × 102.7		3.2047	3.2047
16.14	1.614×10^1		1.2079

Note: *The logarithms of the numerator and denominator are repeated in the fourth column.*

4.4126 – 3.2047 = 1.2079

Antilog 1.2079 = 1.614×10^1. Therefore,

$\frac{743.1 \times 34.8}{15.6 \times 102.7}$ = 16.14.

(c)

No.	Standard form	logarithm	
0.0312 0.00123	3.12×10^{-2} 1.23×10^{-4}	$\bar{2}.4942$ $\bar{4}.0899$	
0.0312 × 0.00123		$\bar{6}.5841$	$\bar{6}.5841$
0.215 0.0802	2.15×10^{-1} 8.02×10^{-2}	$\bar{1}.3324$ $\bar{2}.9042$	-
0.215 × 0.0802		$\bar{2}.2366$	$\bar{2}.2366$
0.0002226	2.226×10^{-4}		$\bar{4}.3475$

Antilog $\bar{4}.3475 = 2.226 \times 10^{-4}$

Exercise 7.7

Evaluate each of the following:

1. $\frac{2.63 \times 8.12}{13.24}$

2. $\frac{5.78}{5.32 \times 6.09}$

3. $\frac{0.48 \times 1.83}{0.042 \times 0.181}$

4. $\frac{350 \times 468}{1143 \times 184}$

5. $\frac{0.33 \times 0.045}{0.29 \times 0.67}$

6. $\frac{13.81 \times 0.31}{0.93 \times 14.27}$

7. $\frac{2.61}{4.83 \times 0.11 \times 7.92}$

8. $\frac{1}{13.68 \times 8.09}$

9. $\frac{34.8 \div 7.362}{1.89 \div 0.0413}$

10. $\frac{5.3 \times 6.94 \div 1.45}{4.71 \div 9.46 \times 7.28}$

Negative characteristics

Great care must be taken when dealing with negative characteristics. Here are some examples on how to handle negative characteristics.

(a) $\bar{2}.53 + \bar{3}.46 = \bar{2} + \bar{3} + 0.53 + 0.46$
$= \bar{5}.99$

(b) $\bar{5}.62 + \bar{6}.71 = (\bar{5} + \bar{6}) + 0.62 + 0.71$
$= -11 + 1.33$
$= \overline{10}.33$

(c) $2.42 - 4.22 = 2 - 4 + 0.4 - 0.22$
$= -2 + 0.28 = \bar{2}.28$

(d) $\bar{5}.81 - \bar{3}.54 = -5 + 0.81$
$\underline{-(-3 + 0.54)}$
$-2 + 0.27$
$= \bar{2}.27$

(e) $\bar{2}.03 - \bar{4}.18 = -3 + 1.03$
$- \underline{(-4+0.18)}$
$1 + 0.85$
$= 1.85$

Exercise 7.8

Simplify the following logarithms:

1. $5.834 + 0.412$
2. $1.001 + 3.121$
3. $3.4 - 4.1$
4. $3.1 - 5.2$
5. $\bar{2}.9 + \bar{1}.1$
6. $\bar{7}.8 + 6.5$
7. $\bar{3}.6 - \bar{1}.4$
8. $\bar{4}.5 - 5.2$
9. $\bar{1}.2 - 2.1$
10. $\bar{5}.3 - \bar{3}.7$

11. $\bar{4}.6 - \bar{7}.7$

12. 6.4 - 7.5

Multiplying and dividing logarithms

Logarithms can be multiplied or divided by integers. Multiplication and division of logarithms with positive characteristics is done in the same way decimals are multiplied or divided by integers.

Example 13

Work out the following multiplications and divisions involving logarithms.

(a) 2.418 ÷ 3

(b) 1.319 × 4

(c) $\bar{2}.645 \times 3$

(d) $\bar{6}.34 \div 2$

(e) $\bar{3}.436 \div 2$

(f) $\bar{1}.312 \div 4$

Solutions

(a) 2.418 ÷ 3 = 0.806

(b) 1.319 × 4 = 5.276

(c) $\bar{2}.645 \times 3 = (-2 + 0.645) \times 3$
$= -6 + 1.935$
$= \bar{5}.935$

(d) $\bar{6}.34 \div 2 = (-6 + 0.34) \div 2$
$= \bar{3} + 0.17$
$= \bar{3}.17$

(e) $\bar{3}.436 \div 2 = (-3 + 0.436) \div 2$

Remember that it is only the characteristic, and not the mantissa, which is negative. Therefore, we re-write 3 so that the negative characteristic is divisible by 2.

Note: -3 = -4 + 1, -6 + 3, -8 + 5, and so on.

Thus, $\bar{3}.436 \div 2 = (-4 + 1 + 0.436) \div 2$
$= (-4 + 1.436) \div 2$
$= -2 + 0.718$
$= \bar{2}.7.18$

(f) $\bar{1}.312 \div 4 = (-4 + 3 + 0.312) \div 4$
$= (-4 + 3.312) \div 4$
$= -1 + 0.828$
$= \bar{1}.828$

Exercise 7.9

Simplify the following logarithms:

1. 3.8 × 2

2. 5.1 ÷ 3

3. $\bar{4}.3 \times 3$

4. $\bar{1}.6 \times 2$

5. $\bar{6}.2 \div 3$

6. $\bar{3}.9 \div 3$

7. $\bar{5}.1 \div 2$

8. $\bar{1}.8 \div 4$

9. $\bar{2}.5 \div 5$

10. $\bar{1}.04 \div 6$

Indices and logarithms

From the laws of indices, we know that $(a^n)^m = a^{nm}$. Therefore, when dealing with logarithms we can use this rule.

Example 14

Use logarithms to evaluate:

(a) $(3.8)^3$

(b) $(0.573)^2$

(c) $\sqrt[3]{16.29}$

(d) $\sqrt[5]{0.035}$

Solutions:

(a) From the tables $3.8 = 10^{0.5798}$. Thus,

$(3.8)^3 = (10^{0.5798})^3$

$= 10^{1.7394}$

Antilog $1.7394 = 5.487 \times 10^1$

$= 54.87$

Note: Log 3.8 = 0.5798 and log $(3.8)^3 = 3 \times 0.5798 = 1.7394$

In tabular form:

No.	Logarithm
3.8 3.8^3	0.5798 0.5798×3
54.87	1.7394

Antilog $1.7394 = 5.487 \times 10$. Therefore, $(3.8)^3 = 54.87$

(b)

No.	Standard form	logarithm
0.573 0.573^2	5.73×10^{-1}	$\bar{1}.7582$ $\bar{1}.7582 \times 2$ $= \bar{1}.5164$
0.3285	3.285×10^{-1}	$\bar{1}.5164$

Antilog $\bar{1}.5164 = 3.285 \times 10^{-1}$. Therefore, $(0.573)^2 = 0.3285$

(c)

No.	Standard form	logarithm
16.29 $\sqrt[3]{16.29}$	1.629×10^1	1.2119 $1.2119 \div 3$ $= 0.4040$
2.534	2.534×10^0	0.4040

$\sqrt[3]{16.29} = 2.534$

(d)

No.	Standard form	logarithm
0.035 $\sqrt[5]{0.035}$	3.5×10^{-2}	$\bar{2}.5490$ $\bar{2}.5490 \div 5$ $= \bar{1}.7098$
0.5126	5.126×10^{-10}	$\bar{1}.7098$

Note: $\frac{\bar{2}.5490}{5} = \frac{\bar{5}+3.549}{5} = \bar{1}.7098$

Antilog $\bar{1}.7098 = 5.126 \times 10^{-1}$.

Therefore, $\sqrt[5]{0.035} = 0.5126$.

Example 15

Use logarithms to evaluate $\sqrt[3]{\frac{0.256 \times 14.7}{8.6 \times 38.5}}$

Solution

No.	Standard form	logarithm	
0.256 14.7	2.56×10^{-1} 1.47×10^{1}	$\bar{1}.4082$ 1.1673	
0.256 × 14.7		0.5755	0.5755
8.6 38.5	 3.85×10^{1}	0.9345 1.5855	-
8.6 × 38.5		2.5200	2.5200
$\sqrt[3]{\frac{0.256 \times 14.7}{8.6 \times 38.5}}$			$\frac{\bar{2}.0555}{3}$ $= \bar{1}.3518$
0.2248	2.248×10^{-1}		$\bar{1}.3518$

Note: $\bar{2}.0555 = \frac{\bar{3}+1.0555}{3}$ $\quad$ $\bar{3}+1=\bar{2}$

$= \bar{1}.3518$

Antilog $\bar{1}.3518 = 2.248 \times 10^{-1}$

Therefore, $\sqrt[3]{\frac{0.256 \times 14.7}{8.6 \times 38.5}} = 0.2248$.

Exercise 7.10

Use logarithms to evaluate:

1. $(7.48)^3$ **2.** $(14.5)^5$

3. $(0.249)^6$ **4.** $(0.018)^3$

5. $(0.0483)^2$ **6.** $\sqrt{310}$

7. $\sqrt[3]{48.5}$

8. $\sqrt[4]{113.2}$

9. $\sqrt{0.491}$

10. $\sqrt[4]{0.0514}$

11. $\sqrt[3]{0.134}$

12. $\sqrt{\frac{0.0189}{2.35}}$

13. $\frac{(0.456)^3}{\sqrt{0.0347}}$

14. $\sqrt{\frac{(0.245)^2 \times (3.14)^3}{0.007 \times 34.1}}$

15. 0.823 × 0.0621

16. $\frac{5.02 \times 13.71}{94.6}$

17. $\frac{\sqrt{12.81}}{33.6}$

18. $\sqrt[5]{0.0137}$

19. $\sqrt[3]{\frac{5.28}{4.093}}$

20. $\sqrt[4]{\frac{0.63^2 \times 0.123}{0.092 \times 14.7}}$

21. $\frac{26.25 \times 0.752}{\sqrt{3.41 \div 6.03}}$

22. $\sqrt{\left(\frac{0.0364 \times 15.6^3}{2.82 \times 46.7}\right)}$

Chapter 8

RATIO, PROPORTION AND PERCENTAGE

Ratio

Quantities of the same kind may be compared in various ways. For example, consider two line segments PQ and RS, where PQ = 8 cm and RS = 12 cm

(a) RS may be said to be 4 cm longer than PQ. Thus we compare the two lengths by giving the difference between their lengths.

(b) Alternatively, the length of PQ may be given as a fraction of RS. Thus:

$$\frac{PQ}{RS}=\frac{8}{12}=\frac{2}{3}$$

(c) The two lengths may also be compared in ratio form thus, PQ:RS = 2:3, read as *ratio of PQ to RS is equal to two to three.*

However, RS:PQ is 12:8 = 3:2, which means PQ:RS ≠ RS:PQ

A ratio involves comparing similar quantities whose units must be the same. Ratios are normally given in their simplest form and have no units of measurements.

Example 8.1

Express the following ratios in their simplest form:

(a) 3 days to 3 weeks (b) sh. 6 to 40 cents.

Solutions

(a) 3 days:3weeks = 3 days : 21days
= 3 : 21
= 1 : 7

(b) sh. 6 : 40 cents = 600 cents : 40 cents
= 600 : 40
= 15 : 1

Example 8.2

Express the following ratios in their simplest forms:

(a) 0.02 : 0.8

(b) $\frac{2}{3}:\frac{4}{5}$

Solutions

(a) *Multiply the decimals by suitable powers of 10 to obtain whole numbers.*
0.02: 0.8 = 0.02 × 100: 0.8 × 100
= 2:80
= 1:40

(b) *Multiply the fractions by the LCM of their denominators to obtain whole numbers.*
$\frac{2}{3}:\frac{4}{5} = \frac{2}{3}\times 15 : \frac{4}{5}\times 15$
= 10:12 = 5:6

When converting the ratio form, a:b, to a fraction, the number on the left-hand side of the ratio becomes the numerator while the number on the right becomes the denominator of the fraction, i.e $\frac{a}{b}$.

Example 8.3

The ratio of the heights of two students is 2:3. If the taller student is 180 cm tall, what is the height of the shorter one?

Solution

Height of shorter student : height of taller student = 2 : 3

Thus, $\frac{\text{height of shorter}}{\text{height of taller}} = \frac{2}{3}$

Height of shorter = $\frac{2}{3}$ × 180 cm

= 120 cm

Example 8.4

Rebecca, Aisha and Musa shared sh. 2,400 in the ratio 3:4:5 respectively.
How much did each receive?

Solution

The total amount was divided into 3 + 4 + 5 = 12 shares.

Thus, Rebecca received $\frac{3}{12}\times$ sh.2,400 = sh. 600

Aisha received $\frac{4}{12}\times$sh.2,400 = sh. 800

Musa received $\frac{5}{12}\times$sh..2,400 = sh. 1,000

Example 8.5

If a:b = 2:3 and b:c = 4:5, find a:c.

Solution

To obtain a single ratio a:b:c;

- multiply each ratio a:b and b:c by an appropriate value so that the share for b, which is common in both ratios, is the same.
- get the LCM of the numbers representing b in both ratios.
- change the ratios to have the value of b as the LCM.

The LCM of 3 and 4 is 12.

a:b = 2:3 = 8:12 (multiply each number in the ratio by 4)
b:c = 4:5 = 12:15 (multiply each number in the ratio by 3)
a:b:c = 8:12:15
Therefore, a:c = 8:15

Exercise 8.1

1. Express the following ratios in their simplest forms.

   (a) 9 cm:48 cm (b) 4 weeks:8 days

(c) sh. 150:sh. 350 (d) 3 cm:3 m

(e) 5 cm^3:5 litres (f) 40 min:1 hour

(g) 12 cm^2:1m2 (h) 50 g:0.8 kg

(i) 300 m:2 km

2. Simplify the following ratios.

(a) 26:52 (b) 1:0.2

(c) 0.3:0.06 (d) 0.4:0.1

(e) $\frac{1}{2}:\frac{1}{4}$ (f) $\frac{3}{4}:5$

(g) 3k:20k (h) $\frac{2}{5}:\frac{3}{4}$

(i) $1\frac{1}{2}:3\frac{1}{4}$ (j) 18:6:2

3. Simplify the following ratios.

(a) 3:6:12 (b) 24:36:54

(c) 0.2:1.6:3.6 (d) $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$

(e) 4p:48p:20p (f) $\frac{1}{2}:2:\frac{1}{5}$

(g) 3-3x:5-5x:1-x (h) $\frac{1}{3}:0.2:3$

4. Find the ratio a:c if:

(a) a:b = 1:2, b:c = 2:5 (b) a:b = 2:5, b:c = 4:1

(c) a:m = 5:1, m:c = $\frac{1}{4}:\frac{1}{2}$ (d) a:n = 3:4, n:c = 6:5

5. There are 40 teachers in a school of 500 students. Find the ratio of student to teacher in its simplest form.

6. The angles of a triangle are in the ratio 2:3:4. Calculate the sizes of the angles.

7. The ratio of girls to boys in a school is 2:3. There are 320 girls. How many boys are there?

8. Divide 1,200 kg in the ratio 2:3:5.

9. A sum of money is divided into two parts in the ratio 5:7. If the larger amount is sh. 6300, find the smaller amount.

10. A metal alloy consists of copper, tin and zinc in the ratio 2:5:3. Find the amount of each constituent in 130 kg of the alloy.

11. Mr. Waako has three sons, Kimuli, Kaggwa and Muwendo, aged 12, 16 and 36 years respectively. He divides sh. 64,000 among them in the ratio of their ages. How much does each receive?

Expressing scale in ratio form

Maps and plans of houses have their scales given in the form 1:n. For instance, the scale of a map is 1:200,000. This means that, 1 cm on the map represents 200,000 cm on the ground. Note that the scale can be written as a ratio 1:200,000 or as a fraction $\frac{1}{200000}$.

The latter is sometimes called the **representative fraction** or RF in short.

Example 8.6

Find the scale of a map in which 36 km is represented by 1.8 cm.

Solution

Scale 1.8 cm : 36 km
= 1.8 cm : 3,600,000 cm (*convert km to cm*)
= 18 : 36,000,000 *(multiply both numbers by 10*)
= 1 : 2,000,000 (*divide both numbers by 18*)

Example 8.7

If the ratio of the scale of a map is 1 : 50,000 and the distance between two points on the map is 8 cm, find the actual distance between the two points.
Solution

Let the actual distance be x cm. Then, 8 cm : x cm = 1 : 50,000
Or x : 8 = 50,000 : 1

$\frac{x}{8} = \frac{50,000}{1}$

x = 8 × 50,000 = 400,000
The actual distance is 400,000 cm or 4 km.

Example 8.8

In a scale drawing, the dimensions of a door are 2 cm by 3 cm. If the scale is 1 : 100, what are the actual measurements of the door in metres?

Solution

1 cm represents 100 cm.
2 cm represents 200 cm = 2 m.
3 cm represents 300 cm = 3 m.

The measurements of the door are 2 m by 3 m.

Exercise 8.2

1. Find the representative fraction in each case:

   (a) 1 cm on the map represents 1 km.

   (b) 0.5 cm on the map represents 500 m.

(c) 200 km is represented by 5 cm on the map.

(d) 5 cm on the map represent 25 km.

(e) 2.4 cm on the map represent 48 km.

2. The scale of a map is 1 : 50,000. Find the length on the map of a road 3 km long in centimeters.

3. An area of 2 cm2 on a map represents 4 km^2 on the ground. Find the representative fraction on the map. Leave the answer correct to 1 significant figure.

4. The measurements of a 2-cm cube were trebled. Find the ratio of the new volume to the original volume then express it in the form 1 : n.

5. A rectangle measures 3 cm by 4 cm in a scale drawing. If 1 cm represents 1 km, find the actual area of the rectangular field.

Changing quantities in ratio

When the value of an item changes, we may want to express the new value in terms of the old value.

Example 8.9

The price of a dress is increased from sh.5000 to sh. 8000. Find the ratio of the price increase.

Solution

New price : old price = 8000 : 5000
= 8 : 5

We say that the price has increased in the ratio 8 : 5.

The new price = $\frac{8}{5}\times$ old price.

Example 8.10

If the price of a television falls from sh. 90,000 to sh. 80,000, what is the ratio of the price decrease?

Solution

New price : old price = 80,000 : 90,000
= 8 : 9

We say, in this case, the price has decreased in the ratio 8 : 9.

New price = $\frac{8}{9}\times$old price

In example 8.10, $\frac{8}{9}$ is called the multiplying factor (number) which gives the new value.

In general, $\frac{\text{new value}}{\text{old value}}$ = multiplying factor.

Example 8.11

(a) Increase 600 in the ratio 6 : 5.

(b) Decrease 40 in the ratio 5 : 8.

Solution

(a) New value = $\frac{6}{5}\times600 = 720$

(b) New value = $\frac{5}{8}\times40 = 25$

Exercise 8.3

1. Increase 240 litres in the ratio:

   (a) 4 : 3 (b) 9 : 5

   (c) $\frac{1}{2}:\frac{1}{4}$ (d) $1\frac{1}{2}:1\frac{1}{4}$

2. Decrease 8 kg in the ratio:

   (a) 1 : 2 (b) 0.7 : 1.6

   (c) $\frac{2}{3}:\frac{3}{4}$ (d) $\frac{5}{8}:\frac{2}{3}$

3. The number of goats in a certain farm has decreased from 500 to 350. In what ratio has it decrease?

4. A shopkeeper increased the price of a radio in the ratio 27 : 20. If the price of the radio was sh. 100,000, find the new price.

5. In a certain primary school the number of pupils decreased in the ratio 6 : 7. If the school now has 720 pupils, what was the original number?

6. Maria reduced the amount of money she used to spend on clothing by $\frac{2}{5}$. In what ratio did she reduce her spending on clothing?

7. A test is marked out of a total of 120 scores. If the scores are to be expressed as percentages, in what ratio must the marks be worked out?

8. A number is increased by $\frac{2}{5}$ of its value and then decreased by $\frac{2}{3}$ of its new value. What is the ratio of the final value to the original value?

9. Kijja used to run 45 km daily. When he got injured he reduced the distance in the ratio 4 : 5. What was his new daily running distance?

10. In a school, sh. 72,000 was spent on beans per day. When the price of beans went up in the ratio 5 : 3, the amount of beans consumed was reduced in the ratio 7 : 8. What was the new daily expenditure on beans?

Proportion

Direct proportion

Two quantities are said to be in direct proportion if they increase or decrease at the same rate. From table 8.1 we notice that the ratios of the number of oranges bought and the cost in shillings are equal. Thus, the cost of oranges is proportional to the number of oranges bought. We say that the cost is **directly proportional** to the number of oranges bought.

Table 8.1

Number of oranges	1	2	3	4	5	6	7	8
Total cost in shillings	5	10	15	20	25	30	35	40

Example 8.12

In a primary school, 4 teachers are needed to cater for 60 students.

(a) How many teachers will be needed for 150 students?

(b) How many students will require 6 teachers?

Solution

Method 1: Unitary method

(a) 60 students need 4 teachers.

1 student needs $\frac{4}{60}$ teachers.

150 students need $\frac{4}{60} \times 150 = 10$ teachers.

(b) 4 teachers will be needed by 60 students.

1 teacher is needed by $\frac{60}{4} = 15$ students.

Therefore, 6 teachers are needed by $15 \times 6 = 90$ students

Method 2: Ratio method

(a) The number of students has increased in the ratio 150 : 60
The number of teachers will also increase in the same ratio since they are directly proportional. Thus
$\frac{150}{60} \times 4 = 10$ teachers

(b) Number of teachers has increased in the ratio 6 : 4
Number of students will also increase in the same ratio, thus
$\frac{6}{4} \times 60 = 90$ students.

Method 3:

(a) Let the number of teachers be n.
$\frac{n}{150} = \frac{4}{60}$
Thus, $n = \frac{4}{60} \times 150 = 10$ teachers.

(b) Let the number of students be m.

$\frac{m}{6} = \frac{60}{4}$

$m = \frac{60}{4} \times 6 = 90$ students.

In general, if two quantities, p and q, are directly proportional and one increases (or decreases) in a particular ratio then the other increases (or decreases) in the same ratio.
Thus, p : q or q : p is constant.
In a situation like this, where the ratio of two variables is constant, we say that one quantity is *directly proportional* to the other. Thus, p is directly proportional to q, or, p varies directly as q.
The symbol $\propto$ is used for 'is proportional to'. i.e. $p \propto q$.
This relationship may also be written as an equation, $p = kq$, where k is a constant.
If $y \propto x$ then a constant, k, can be found so that $y = kx$.

Example 8.13

Given that $a \propto b$ and a = 12 when b = 4, find
(a) the relationship between a and b as an equation;
(b) a when b = 9;
(c) b when a = 32.

Solution

$a \propto b$, therefore $a = kb$ or $\frac{a}{b} = k$, where k is a constant.

(a) if a = 12 when b = 4
$\frac{12}{4} = k$. Therefore, k = 3
Thus $\frac{a}{b} = 3$ i.e. a = 3b

(b) If b = 9 and a = 3b
$= 3 \times 9$
Therefore, a = 27.

(c) If a = 32 and a = 3b
$\Rightarrow \quad 32 = 3b$
$\Rightarrow \quad b = \frac{32}{3} = 10\frac{2}{3}$

Example 8.14

Given $y \propto x^2$ and y = 12 when x = 2, find
(a) y when x = 5
(b) x when y = 15.

Solution

$y \propto x^2$

Therefore, $y = kx^2$ (k is a constant)

y = 12 when x = 2

Therefore, $12 = 4k \Rightarrow k = 3$.

Therefore, $y = 3x^2$

(a) If $x = 5$ and $y = 3x^2$,

$y = 3(5)^2$

Therefore, $y = 3 \times 25 = 75$

(b) If $y = 15$ and $y = 3x^2$,

$15 = 3x^2$

$5 = x^2$

Therefore, $x = \pm\sqrt{5}$

Example 8.15

In the following table *b* is directly proportional to *a*.
(a) Find an equation connecting *b* and *a*.
(b) Find the missing numbers in the table.

a	1	2	3		10
b	7	14		49	

Solution

$b \propto a \Rightarrow b = ka$ (where *a* is a constant)

(a) From the table, when $a = 1$, $b = 7$

$\Rightarrow 7 = k.1$

Therefore, $k = 7$

Hence, $b = 7a$ is the equation connecting *b* and *a.*

(b) If $a = 3$ and $b = 7a$, then $b = 7 \times 3 = 21$

If $b = 49$ and $b = 7a$

$\Rightarrow 49 = 7a$

$\Rightarrow a = \frac{49}{7} = 7$

If $a = 10$ and $b = 7a \Rightarrow b = 7 \times 10 = 70$.

Example 8.16

The area of a circle varies directly as the square of its radius. If an area of a circle of radius 7 cm is 154 cm^2, what is the
(a) area of a circle of radius 14 cm;
(b) effect on the area of doubling the radius?

Solution

Let A be the area of the circle and r its radius. Then

$A \propto r^2$

$\Rightarrow A = kr^2$ (k is a constant)

When r = 7, A = 154 $\Rightarrow$ 154 = k.49

Therefore, k = $\frac{154}{49}$ or $\frac{22}{7}$, so that A = $\frac{22}{7}$ r^2

(a) If r = 14 and A = $\frac{22}{7}$ r^2

$\Rightarrow$ A = $\frac{22}{7}\times(14)^2$ = 616 cm^2

(b) When radius is doubled, i.e. 2r, then A^1 = $\frac{22}{7}(2r)^2$ where, A^1 is the new area.

$\Rightarrow$ $A^1 = 4(\frac{22}{7}r^2) = 4A$

Thus, the area is increased 4 times as much.

Exercise 8.4

1. $y \propto x$. If y = 6 when x = 2, find
   (a) y when x = 7; (b) x when y = 42.
2. $y \propto x$. If y = 12 when x = 3, find
   (a) y when x = 8; (b) x when y = 15.
3. $y \propto x$. If y = $\frac{1}{2}$ when x = 4, find
   (a) y when x = 12; (b) x when y = $\frac{1}{8}$
4. $y \propto x$. If y = -5 when x = 10, find
   (a) y when x = -15; (b) x when y = 8
5. $y \propto x^2$. If y = 8 when x = 2, find
   (a) y when x = 3; (b) x when y = 32.
6. $y \propto x^2$. If y = 2 when x = 2, find
   (a) y when x = 4; (b) x when y = $12\frac{1}{2}$
7. $y \propto \sqrt{x}$. If y = 27 when x = 9, find
   (a) y when x = 9; (b) x when y = 98.
8. Given the cost of material is directly proportional to the length and that 2 metres of material costs sh. 7,000, find the cost of 5 metres.
9. If the cost of books varies directly as the number bought and if 5 books cost sh. 2,500, find the cost of 9 books.
10. Travelling at a constant speed, the distance traveled by a car is directly proportional to the time taken. If 190 km is traveled in 4 hours, find
    (a) the distance traveled in 10 hours;
    (b) the time taken to travel 285 km.
11. At constant speed, distance traveled varies directly as time. If a man walks 32 km in 5 h, how far would he have walked in 3 h at the same constant speed?
12. The extension of a spring varies as the weight attached to the spring. If a weight of 10 g produces an extension of 2.8 cm find
    (a) the extension produced by a weight of 16 g;
    (b) the weight which would produce an extension of 4.2 cm;

Inverse proportion

If one quantity increases while the other decreases at the same rate, or one of the quantities decreases while the other increases at the same rate, such quantities are said to be **inversely proportional**. Consider, for example, the time a cyclist would take to cover a distance of 48 km at various speeds. See table 8.2.

Table 8.2

Speed (v) in km/h	4	6	8	12	16
Time taken (t) in hours	12	8	6	4	3

As the speed increases, the time taken for the journey decreases. This is an example of inverse proportion. Speed multiplied by time always has the same value, that is, speed × time = 48 or vt = 48
In general, the product of two inversely proportional quantities is a constant.
The statement 'v is inversely proportional to t', is normally written as

$v \propto \frac{1}{t}$ or $vt = k$ where k is a constant.

Example 8.17

Time (T) varies inversely as speed (S). If a journey takes 5 hours when travelling at a speed of 60 km/h, find

(a) the time for the journey if the speed is 90 km/h;
(b) the speed at which a car must travel if it is to complete the journey in 3 hours.

Solution

$T \propto \frac{1}{S}$ i.e. $TS = k$ (k is a constant)
If T = 5 when S = 60
TS = 5 × 60 = 300
$\therefore$ k = 300

(a) If S = 90 and TS = 300
T × 90 = 300
$\therefore T = \frac{300}{90} = 3\frac{1}{3}$ hours.

(b) If T = 3 and TS = 300
$\Rightarrow 3 \times S = 300$
S = 100 km/h

Example 8.18

y varies inversely as the square of x. If y = 5 when x = 2 find

(a) y when x = 4
(b) x when y = 4

Solution

$y \propto \frac{1}{x^2}$

$x^2y = k$ (k is constant)

$2^2 \times 5 = k$

$k = 20$

$\therefore x^2y = 20$

(a) If $x = 4 \Rightarrow 16y = 20$

$\Rightarrow y = \frac{20}{16} = \frac{5}{4}$

(b) If $y = 4 \Rightarrow 4x^2 = 20$

$\Rightarrow x^2 = 5$

$\Rightarrow x = \pm\sqrt{5}$

Example 8.19

A farmer has enough feed to last his 45 cows 30 days. If he buys 5 more cows, how long will the feed last?

Solution

Unitary method

45 cows can be fed for 30 days

1 cow can be fed for 30 × 45 days

50 cows can feed for $\frac{30 \times 45}{50} = 27$ days

Thus, if 5 cows are added to the herd, the feed will last for 27 days.

Constant product method

Let d be the required number of days.

Then, 50 × d = 45 × 30 = 1,350

$d = \frac{1350}{50} = 27$

Exercise 8.5

1. Write as an equation
   (a) y varies inversely as x;
   (b) p varies inversely as q;
   (c) m varies inversely as the square of n
   (d) r varies inversely as the square of s;
   (e) a varies inversely as the cube of b;
   (f) b varies inversely as the square root of c
2. y varies inversely as x. If y = 6 when x = 6, find
   (a) y when x = 12;
   (b) x when y = 9.
3. a varies inversely as b. If a = -5 when b = 4, find
   (a) a when b = -10
   (b) b when a = 7.

4. y varies inversely as x^2. If $y = 10$ when $x = 2$, find
   (a) y when $x = 4$;
   (b) x when $y = 4$.
5. Time varies inversely as speed for a given journey. If it takes 5 hr to complete the journey when travelling at 80 km/h, how long would the same journey take if the speed were increased to 100 km/h?

6. The length of material which can be bought for a given amount varies inversely as the price. If 12 m can be bought when the price is sh. 2,500 per metre, how much can be bought for the same total cost when the price is sh. 3,000 per metre?

7. A bus uses 40 litres of fuel to travel 100 km. How much fuel will be used on a journey of 68 km?

8. Five pipes of equal sizes can fill a tank in 40 minutes. How long would four of these pipes take to fill the same tank?

9. It takes two hours for a car to cover a distance at 60 km/h. How long will it take to cover the same distance at 80 km/h?

10. A family of 6 has enough food to last 12 days. How long will the food last if the family receives 3 visitors who are to stay for 10 days?

11. A mass choir is to be split into groups for training. If it is split into groups of 8, there will be 15 groups.
    (a) How many groups of 10 would there be?
    (b) If only 6 trainers are available, how many members will be in each group?
12. A class of 30 students uses 75 pencils in a term. If the number of students is reduced to 24, how many pencils are likely to be used in a term?

13. A school has enough money to buy 250 books that cost sh. 15,000 each. How many books costing sh. 10,500 can be bought instead?

14. Given that a is directly proportional to b, copy and complete the following table.

a	6	8	14	28	38	-
b	-	-	21	-	-	40

15. A class room was arranged for a meeting for 84 participants such that there were four rows of chairs. A different arrangement was suggested using the same number of chairs in 7 rows. How many chairs per row were there in the second arrangement?

16. Given that x and y are inversely proportional, copy and complete the following table.

x	6	8	-	48	-	144
y	-	3	2.4	-	216	-

17. A contractor estimates that he can fence an estate in 9 days if he employs 4 people. Will 6 people be enough to complete the work in 5 days?

18. A group of girls finished a piece of work in 10 days. How many extra girls are needed to complete the same work in 8 days?

Percentage

Percentage is a fraction with the denominator of 100. *Per cent* means *in each hundred* and is denoted as %. For example, 34 per cent is written as 34% and it means 34 out of every 100.

Thus, 34% = $\frac{34}{100}$. Similarly, 25% = $\frac{25}{100}$ and 150% = $\frac{150}{100}$.

Converting percentage to fraction

Example 8.20

Convert the following percentages to fractions.

(a) 20% (b) $45\frac{1}{2}\%$

Solutions

(a) $20\% = \frac{20}{100} = \frac{1}{5}$

(b) $45\frac{1}{2}\% = \frac{45\frac{1}{2}}{100} = \frac{91}{2} \div \frac{100}{1} = \frac{91}{200}$

Converting fractions to percentages

A fraction can be expressed as a percentage by multiplying it by 100.

Example 8.21

Convert the following fractions to percentages.

(a) $\frac{1}{8}$ (b) 0.625

(c) $3\frac{1}{2}$ (d) 3.25

Solutions

(a) $\frac{1}{8} = \frac{1}{8} \times 100\% = 12.5\%$

(b) $0.625 = 0.625 \times 100\% = 62.5\%$

(c) $3\frac{1}{2} = \frac{7}{2} \times 100\%$

$= 3.5 \times 100\% = 350\%$

(d) $3.25 = 3.25 \times 100\% = 325\%$

Note that decimals and fractions that are greater than 1 convert to percentages that are greater than 100%.

Quantities as percentages

To express on quantity as a percentage of another quantity
(a) express in the same units
(b) write as a fraction
(c) multiply by $\frac{100}{1}$ and add % sign.

Example 8.22

Express 28 as a percentage of 40.

Solution

Converting to a percentage,

$$\frac{28}{40} = \frac{28^{7}}{{}_{1}40} \times \frac{100}{1}\%$$
$$= 70\%$$

Example 8.23

Express 20 cm as a percentage of 5 m.

Solution

5 m = 500 cm
Thus 20 cm as a fraction of 500 cm = $\frac{20}{500}$ and 20 cm as a percentage of 500 cm = $\frac{20}{500} \times \frac{100}{1}\% = 4\%$

Percentage of quantities

To find a percentage of a given quantity
(a) Write the percentage as a fraction
(b) multiply the fraction by the given quantity.

Example 8.24

Find
(a) 5% of 25 (b) $33\frac{1}{3}$% of sh. 12,000

Solution

(a) 5% of 25 means $\frac{5}{100} \times 25 = \frac{5}{4}$
$= 1\frac{1}{4}$

(b) $33\frac{1}{3}$% of sh. 12,000 = $\frac{33\frac{1}{3}}{100} \times 12{,}000$
$= \frac{100}{300} \times 12{,}000$
= sh. 4,000

Example 8.25

If 20% of a class is boys and there are 5 boys, how many students are in the class?

Solution

20% is 5

1% is $\frac{5}{20}$

100% is $\frac{5}{20} \times 100 = 25$

Therefore, there are 25 students in the class.

Percentage increase or decrease

Percentage increase or decrease are of significant importance in real life. A Shopkeeper who wants to reduce the prices of goods finds it practical to reduce the prices on the basis of a percentage.

Example 8.26

The price of a T.V set costing sh. 230,000 goes up by 12%. What is the new price?

Solution

The increase = 12% of sh. 230,000
= 0.12 × sh. 230,000
= sh. 27,600
The new price is 230,000 + sh. 27,600 = sh. 257,600

Alternatively, the new price is (100% + 12%) of 230,000
= 112% of sh. 230,000
= 1.12 × sh. 230,000
= sh. 257,000

Example 8.27

The price of a ream of paper that costs sh. 7000 is reduced by 10%. What is the new price?

Solution

The price decrease = 10% of sh. 7,000
= $\frac{10}{100} \times$ sh.7,000
= sh. 700
The new price is sh. (7,000 – 700) = sh. 6,300

Alternatively, the new price is (100% - 10%) of sh. 7,000
= 90% of sh. 7,000
= 0.9 × sh. 7,000
= sh. 6,300

Example 8.28

In a sale, a shirt that cost sh. 8,000 is sold at sh. 6,800. What is the percentage decrease in price?

Solution

The percentage decrease = $\frac{8000-6800}{8000} \times 100\%$

$= \frac{1200}{8000} \times 100\%$

$= 15\%$

Example 8.29

The population of a town increased from 200,000 people to 350,000 people. What is the percentage increase?

Solution

Percentage increase = $\frac{350,000-200,000}{200,000} \times 100\%$

$= \frac{150,000}{200,000} \times 100\%$

$= 75\%$

In general, percentage = $\frac{\text{increase/decrease}}{\text{original value}} \times 100\%$

Exercise 8.6

1. In a by-election, 18,500 votes were cast. If the winning candidate received 30% of the total votes cast, how many votes were cast in favour of the winning candidate?
2. In a mathematics test marked out of 50 marks, Mbatya scored 27%. How many marks did she score?
3. Increase the stated quantities by the percentages given in brackets.
   (a) 200 (14%) (b) sh. 5,000 (0.23%)
4. Decrease the stated quantities by the percentages given in brackets.
   (a) 400 litres (12.5%) (b) 60 kg (35%)
5. Twenty-four per cent of the fruits in a bag are oranges, 48% are apples and the rest are pears. What percentage of the fruits are pears?
6. Prices of commodities in a shop are reduced by13%. How much will a lamp, previously costing sh. 3,200, be sold for?
7. A pond has 1,200 fish. If the number of fish increases by $8\frac{1}{2}$% what will be the new population of fish in the pond?

8. A company increased salaries by 20%. If Kenneth and Hadijja were earning sh. 130,000 and 650,000 per month respectively, by how much were their salaries increased?

9. A carpet is laid in a room that measures 3.5 m by 5.2 m. The carpet covers 60% of the floor. Find the area of the carpet.

10. Mwanga and Leah each has a salary of sh. 540,000. Mwanga is given a rise of 12% in one year and the same percentage increase in the next year. Leah is offered 14% increase for the two years. Who is offered a better deal and by how much over the two years?

11. In a school, 64% of the pupils study Physics and 648 do not. How many pupils are there in the school?

12. A trader raised the price of an article by 25%. When she realized that nobody was buying it, she lowered the new price by 25%. If the old price was sh. 2400, what is the new price after the reduction?

13. The radius of a circle is increased by 5%. What is the percentage increase in the
    (a) circumference?
    (b) area?

14. The ratio of boys to girls in a class is 2:3. What is the percentage of girls in the class?

15. A businesswoman bought 800 eggs at sh. 150 each. If $6\frac{1}{4}$% got broken and were thrown away, for how much should she sell each egg in order to make a profit of 50%?

16. Express the following as fractions.
    (a) 70% (b) 135% (c) 15.75%

17. Express the following as decimals.
    (a) 60% (b) 175% (c) 8.5%

18. Express the following as percentages.
    (a) $\frac{17}{100}$ (b) $\frac{36}{25}$ (c) 0.45

19. A salesman receives a commission of sh. 4,000 if she sells goods worth sh. 400,000. Find the ratio of the commission to the sales.

20. On a map of scale 1 : 50,000, the distance between two points is 25.6 cm. Find the actual distance in kilometers.

21. Sixteen people can make 800 pots in 9 days. How long will 15 people take to make 1,000 pots?

22. Twenty people working $7\frac{1}{2}$ hours a day can finish a piece of work in 21 days. Find how many hours a day must 45 people work to finish the work in 7 days.

23. The dimensions of a rectangle measuring 8 cm by 2.5 cm are increased by 20%. Find the percentage increase in the area.

24. Express
    (a) 5 cm as a percentage of 20 cm
    (b) 30 cm as a percentage of 3 m
    (c) 1.20 m as a percentage of 40 cm
    (d) 45 kg as a percentage of 25 kg
    (e) 3 kg as percentage of 500 g

25. On a test a student scored 25 out of a possible 40 marks. What percentage mark did he receive?

Chapter 9

QUADRATIC EXPRESSIONS AND EQUATIONS

Expanding algebraic expressions

Consider the product of (a + b) and (c + d). This product can be written as (a + b)(c + d).
Suppose we let (c + d) = u
Then, (a + b)(c + d) = (a + b) u
= au + bu
= a(c + d) + b(c + d)
= ac + ad + bc + bd
We see that each term in the first brackets is multiplied by each term in the second brackets.

Example 9.1

Expand (3x + 2)(5 + 4y).

Solution

Let (5 + 4y) = u

(3x + 2)(5 + 4y) = (3x + 2) u
= 3xu + 2u
= 3x(5 + 4y) + 2(5 + 4y)
= 15x + 12xy + 10 + 8y
$\therefore$ (3x + 2)(5 + 4y) = 15x + 12xy + 10 + 8y.

Example 9.2

Expand (2a – 5)(b + 9).

Solution

(2a - 5)(b + 9) = 2a(b + 9) – 5(b + 9)
= 2ab + 18a – 5b – 45

We have multiplied each term in the second brackets by -5 and not just 5. Terms in an expression are identified with the signs before them.

Example 9.3

Expand $(x + \frac{3}{5})(y - \frac{1}{4})$

Solution

$(x + \frac{3}{5})(y - \frac{1}{4}) = xy - \frac{1}{4}x + \frac{3}{5}y - \frac{3}{20}$

Exercise 9.1

Expand.

1. (x + 1)(4 + y)

2. (6a + 1)(c + 3)

3. (2a + b)(c + 3d)

4. (x - y)(4 - z)

5. $(a - 3d)(4c - b)$ 6. $(8x + y)(a + 1)$

7. $(2u - v)(3x - y)$ 8. $(p + q)(s - t)$

9. $(x + y)(c - 5)$ 10. $(x + 1)(y - 1)$

11. $(x + \frac{1}{2})(y + \frac{1}{4})$ 12. $(u + w)(a + 2)$

13. $(5a + 4)(b - 3)$ 14. $(x + y)(\frac{1}{2}a - c)$

Quadratic expressions

Consider the product of $(x + 3)(x + 4)$.

$$\begin{aligned}(x + 3)(x + 4) &= x(x + 4) + 3(x + 4)\\ &= x^2 + 4x + 3x + 12\\ &= x^2 + 7x + 12\end{aligned}$$

This kind of an expression where the highest power is 2 is called a **quadratic expression**. Most quadratic expressions have three distinct terms known as the quadratic term (x^2), the linear term ($7x$) and the number term or constant term (12).

A quadratic expression must have the quadratic term which should also be the term with the highest power of the unknown letter. For example, $x^2 + 3x + 1$, $a^2 + 2a$, $y^2 - 2y + 1$, $c^2 - 4$, e.t.c. but $x^3 - 2x$, $2x^3 - x^2 + 6$, $4x + 8$ are not.

Example 9.4

Expand and simplify:

(a) $(2x + 1)(x + 3)$ (b) $(x - 5)(x - 2)$,

(c) $(2a - 3)(5a - 4)$.

Solutions

(a) $$\begin{aligned}(2x + 1)(x + 3) &= 2x(x + 3) + 1(x + 3)\\ &= 2x^2 + 6x + x + 3\\ &= 2x^2 + 7x + 3\end{aligned}$$

(b) $$\begin{aligned}(x - 5)(x - 2) &= x(x - 2) - 5(x - 2)\\ &= x^2 - 2x - 5x + 10\\ &= x^2 - 7x + 10\end{aligned}$$

(c) $$\begin{aligned}(2a - 3)(5a - 4) &= 2a(5a - 4) - 3(5a - 4)\\ &= 10a^2 - 8a - 15a + 12\\ &= 10a^2 - 23a + 12\end{aligned}$$

Exercise 9.2

Expand and simplify.

1. $(x + 3)(x + 1)$ 2. $(x + 2)(x - 4)$

3. $(x - 7)(x + 3)$ 4. $(x - 4)(x - 3)$

5. $(a + 8)(a - 3)$ 6. $(p + 2)(p - 5)$

7. $(x - 5)(x + 4)$ 8. $(x + 5)(x + 4)$

9. $(x + 2)(5x + 3)$

10. $(3x + 2)(x - 4)$

11. $(2x - 7)(4x - 3)$

12. $(x +3)(x + 3)$

13. $(a - 4)(a - 4)$

14. $(x + 7)(x - 7)$

15. $(5t + 3)(3t + 2)$

16. $(2 - x)(3 - x)$

17. $(3 + p)(5 - p)$

18. $(4 - 2y)(1 - 3y)$

19. $(3x + 1)(8 - 2x)$

20. $(5 - 3x)(2 - 4x)$

The quadratic identities

1. *Show that* $(a + b)(a + b) = a^2 + 2ab + b^2$.

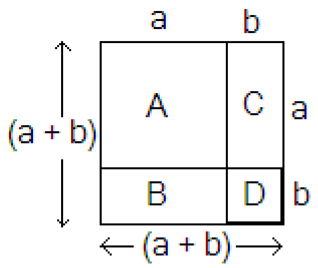
The product $(a + b)(a + b)$ can be written as $(a + b)^2$. Thus,

$(a + b)^2 = (a + b)(a + b)$

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

This product is illustrated in the figure below.



Consider a square with side (a + b) units. The square is divided into:

(a) square A of side a,
(b) square D of side b,
(c) rectangle B of sides a and b,
(d) rectangle C of sides a and b.

Area of the big square = $(a + b)^2$

Area of: square A = a^2

rectangle B = ab

rectangle C = ab

square D = b^2

Rectangles B and C are identical.

The area of A + B + C + D = $a^2 + ab + ab + b^2$

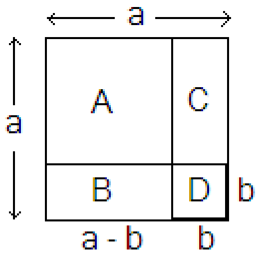
Therefore, $(a + b)^2 = a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

2. *Show that* $(a - b)(a - b) = a^2 - 2ab + b^2$

Consider a square of side a units. The square can be divided into two squares: one of side (a - b) and another of side b; and two identical rectangles of side b and (a - b). That is:

(a) square A has sides (a - b),
(b) rectangles B and C have sides b by (a - b),
(c) square D has side b.

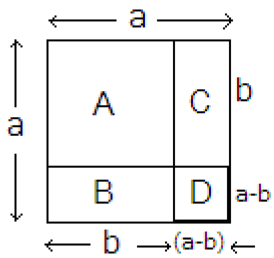


Area of the big square = a^2 ....(i)
Area of square A = $(a - b)(a - b)$
= $(a - b)^2$
Area of rectangle B = $(a - b)b$
= $ab - b^2$
Area of rectangle C = $(a - b)b$
= $ab - b^2$
Area of square D = b^2
Total area = $(a - b)^2 + ab - b^2 + ab - b^2 + b^2$
= $(a - b)^2 + 2ab - b^2$ ....(ii)
Equating (i) and (ii) gives:
$(a - b)^2 + 2ab - b^2 = a^2$
Therefore, $(a - b)^2 = a^2 - 2ab + b^2$.

Note: $(a + b)^2 \neq a^2 + b^2$ and $(a - b)^2 \neq a^2 - b^2$

3. *Show that* $(a + b)(a - b) = a^2 - b^2$

Consider a square of side a units. See figure below.



The square is divided into:
(a) square A of side b,
(b) square D of side (a - b),
(c) rectangles B and C of sides b by a - b.

Area of the big square = a^2 ....(i)
Area of square A = b^2
Area of rectangle B = $(a - b)b$
Area of rectangle C = $(a - b)b$
Area of square D = $(a - b)^2$
Total area = $(a - b)b + (a - b)b + (a - b)^2 + b^2$
= $2(a - b)b + (a - b)^2 + b^2$..........(ii)
Equating (i) and (ii) gives:
$2(a - b)b + (a - b)^2 + b^2 = a^2$
Therefore, $2b(a - b) + (a - b)^2 = a^2 - b^2$
This means, $(a - b)[2b + (a - b)] = a^2 - b^2$, a - b is common.
That is, $(a - b)(2b + a - b) = a^2 - b^2$,
Therefore, $(a + b)(a - b) = a^2 - b^2$

Expanding, $(a + b)(a - b) = a(a - b) + b(a - b)$
$= a^2 - ab + ab - b^2$
$= a^2 - b^2$

The expression, $(a + b)(a - b)$, is called **the difference of two squares**.

The three important identities can thus be summarized as:

(a) $(a + b)^2 = a^2 + 2ab + b^2$
(b) $(a - b)^2 = a^2 - 2ab + b^2$
(c) $(a + b)(a - b) = a^2 - b^2$

Example 9.5

Expand and simplify:

(a) $(x + 4)^2$ (b) $(x - 5)^2$
(c) $(x + 3)(x - 3)$ (d) $(2x + 1)^2$
(e) $(3x + 2)((3x - 2)$

Solutions

(a) $(x + 4)^2 = (x + 4)(x + 4)$
$= x(x + 4) + 4(x + 4)$
$= x^2 + 4x + 4x + 16$
$= x^2 + 8x + 16$

(b) $(x - 5)^2 = (x - 5)(x - 5)$
$= x(x - 5) - 5(x - 5)$
$= x^2 - 5x - 5x + 25$
$= x^2 - 10x + 25$

(c) $(x + 3)(x - 3) = x(x - 3) + 3(x - 3)$
$= x^2 - 3x + 3x -9$
$= x^2 - 9$

Note: $(x^2 - 9 = x^2 - 3^2)$

(d) $(2x + 1)^2 = (2x + 1)(2x + 1)$
$= 2x(2x + 1) + 1(2x + 1)$
$= 4x^2 + 2x + 2x + 1$
$= 4x^2 + 4x + 1$

(e) $(3x + 2)(3x - 2) = 3x(3x - 2) + 2(3x - 2)$
$= 9x^2 - 6x + 6x - 4$
$= 9x^2 - 4$

Note: $9x^2 - 4 = (3x)^2 - 2^2$.

Exercise 9.3: Expand.

1. $(x + 1)^2$ **2.** $(x + 2)^2$
3. $(a - 5)^2$ **4.** $(p + q)^2$
5. $(a + 6)^2$ **6.** $(b - 8)^2$
7. $(x - y)^2$ **8.** $(x + 1)(x - 1)$
9. $(y - 9)(y + 9)$ **10.** $(y - x)(y + x)$
11. $(y - a)(y + a)$ **12.** $(3x + 4)^2$

13. $(2a - 7)(2a + 7)$

14. $(5x + 3)^2$

15. $(4x - 1)^2$

16. $(7a - 3)^2$

17. $(2x - 1)^2$

18. $(3 - 2x)^2$

19. $(c - ax)^2$

20. $(x - 1)^2$

21. $\left(u + \frac{1}{4}\right)^2$

22. $\left(a - \frac{3}{4}\right)^2$

23. $\left(2x + \frac{1}{3}\right)^2$

24. $\left(\frac{1}{2}x - 3\right)^2$

Factorizing expressions

If the product of 5 and 7 is 35, then 5 and 7 are factors of 35.

In algebra, letters represent numbers. Therefore, we can extend the idea of factors to algebraic expressions. For example, a and b are factors of ab and 2, x and y are factors of 2xy. Also, given that $5(a - b) = 5a - 5b$, then, 5 and $(a - b)$ are factors of $5a - 5b$.

In order to find the factors of an expression such as $10x^2 + 15x$, we look for the factors that are common in both terms. The common factors of $10x^2$ and 15x are 5 and x.

Thus, $10x^2 + 15x = (5x \times 2x) + (5x \times 3)$
$= 5x(2x + 3)$.

Therefore, the factors of $10x^2 + 15x$ are 5, x and $(2x + 3)$. This means, $10x^2 + 15x$ can be factorized as $5x(2x + 3)$.

The process of finding the factors of an expression is called **factorization**. This is the reverse of expansion.

Consider $(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$
$= x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$.

The expressions $(x + 2)$ and $(x + 3)$ are the factors of $x^2 + 5x + 6$.

Similarly:

(a) $(x + 3)(x + 4) = x(x + 4) + 3(x + 4)$
$= x^2 + 4x + 3x + 12$
$= x^2 + 7x + 12$.

(b) $(x + 2)(x + 7) = x(x + 7) + 2(x + 7)$
$= x^2 + 7x + 2x + 14$
$= x^2 + 9x + 14$

(c) $(x + a)(x + b) = x(x + b) + a(x + b)$
$= x^2 + bx + ax + ab$
$= x^2 + (a + b)x + ab$

The last expression shows the relationship between the terms of the quadratic expression and its factors. Thus, a constant term *ab* of a quadratic expression is the product of the number terms *a* and *b*. whereas, the coefficient of x is the sum of *a* and *b*.

Example 9.6

Factorize $x^2 + 10x + 21$.

Solution

Compare $x^2 + 10x + 21$ and (x + __)(x + __), considering
$x^2 + (a + b)x + ab = (x + a)(x + b)$.
This means $x^2 + 10x + 21 = x^2 + (a + b)x + ab$.
We need two numbers *a* and *b* such that $a + b = 10$ and $ab = 21$
Clearly, *a* and *b* are factors of 21 whose sum is 10. These are 3 and 7.
Rewriting $x^2 + 10x + 21$ as $x^2 + (3 + 7)x + 21$ gives $x^2 + 3x + 7x + 21$
$= x(x + 3) + 7(x + 3)$
$= (x + 3)(x + 7)$.
Therefore, $x^2 + 10x + 21 = (x + 3)(x + 7)$.

Example 9.7

Factorize $x^2 + 7x + 12$.

Solution

We need two numbers a and b such that a + b = 7 and ab = 12. These numbers are 3 and 4.
Thus, $x^2 + 7x + 12 = x^2 + (3 + 4)x + 12$
$= x^2 + 3x + 4x + 12$
$= x(x + 3) + 4(x + 3)$
$= (x + 3)(x + 4)$

Example 9.8

Factorize x2 - 6x + 8

Solution

Let, $x^2 - 6x + 8 = (x + a)(x + b)$.
Thus. $X^2 - 6x + 8 = x^2 + (a + b) + ab$.
Then, $(a + b) = -6$ and $ab = 8$
Two numbers which add up to -6 and whose product is 8 are -2 and -4.
So, $x^2 - 6x + 8 = x^2 + (-2 + -4)x + 8$
$= x^2 - 2x - 4x + 8$
$= x(x - 2) - 4(x - 2)$
$= (x - 2)(x - 4)$.

Example 9.9

Factorize $x^2 - x - 30$

Solution

Remember that $x^2 - x - 30 = x^2 - 1x - 30$
Let $x^2 - x - 30 = (x + a)(x + b)$
$= x^2 + (a + b)x + ab$.
Thus, $(a + b) = -1$ and $ab = -30$. 5 and -6 satisfy both of these equations.
So, $x^2 - x - 30 = (x + 5)(x - 6)$

Example 9.10

Factorize $x^2 + 3x - 28$.

Solution

Let $x^2 + 3x - 28 = (x + a)(x + b) = x^2 + (a + b)x + ab$
Thus, $(a + b) = 3$ and $ab = -28$.
The factors of -28 whose sum is 3, are 7 and -4.
So, $x^2 + 3x - 28 = (x + 7)(x - 4)$

Example 9.11

Factorize:
(a) $x^2 + 12x + 36$ (b) $x^2 - 8x + 16$

Solutions

(a) Let $x^2 + 12x + 36 = (x + p)(x + q) = x^2 + (p + q)x + pq$
Thus, $p + q = 12$ and $pq = 36$.
The factors of 36 whose sum is 12, are 6 and 6.
So, $x^2 + 12x + 36 = (x + 6)(x + 6) = (x + 6)^2$

(b) Let, $x^2 - 8x + 16 = (x + p)(x + q) = x^2 + (p + q)x + pq$
Thus, $p + q = -8$ and $pq = 16$
The factors of 16 whose sum is -8, are -4 and -4.
So, $x^2 - 8x + 16 = (x - 4)(x - 4) = (x - 4)^2$

Note: $x^2 + 16x + 36$ *and* $x^2 - 8x + 16$ *are perfect squares because their factors are identical.*

Exercise 9.4

Factorize:

1. $x^2 + 7x + 10$ **2.** $x^2 + 13x + 42$

3. $x^2 + 9x + 14$ **4.** $x^2 + 2x - 15$

5. $x^2 + x - 12$ **6.** $p^2 - 10p + 12$

7. $a^2 - 2a - 35$ **8.** $d^2 + 5d - 36$

9. $v^2 - 14v + 45$ **10.** $u^2 - u - 56$

11. $x^2 + 9x + 8$ **12.** $n^2 - 10n + 25$

13. $x^2 - 2x + 1$ **14.** $c^2 - c - 2$

15. $y^2 + 14y + 24$ **16.** $w^2 + 5w - 6$

17. $9 - 6r + r^2$ **18.** $49 + 14t + t^2$

19. $4 + 4k + k^2$ **20.** $x^2 + 11x - 26$

The difference of two squares

Earlier in this chapter, we learnt that $(a + b)(a - b) = a^2 - b^2$. Thus, the difference of the squares of two numbers is equal to the product of their sum and their difference. The factors of $a^2 - b^2$ are $(a + b)$ and $(a - b)$.

In order to factorize the difference of two squares, it is important to rewrite the expression so that the squares are clearly seen. For example:

(a) $a^2 - 9b^2 = a^2 - (3b)^2$
$= (a - 3b)(a + 3b)$,

(b) $1 - 16x^2 = 1^2 - (4x)^2$
$= (1 + 4x)(1 - 4x)$,

(c) $4x^2 - 25y^2 = (2x)^2 - (5y)^2$
$= (2x + 5y)(2x - 5y)$.

However, some expressions which involve the difference of squares require us to find the common factors first.

Example 9.12

Factorize:

(a) $7x^2 - 7y^2$ (b) $3x^2 - 75y^2$

Solutions

(a) Both terms have a common factor, 7. Therefore factorize as follows:
$7x^2 - 7y^2 = 7(x^2 - y^2)$
$= 7(x + y(x - y)$

(b) 3 is a common factor in the two terms.
Thus, $3x^2 - 75y^2 = 3(x^2 - 25y^2)$
$= 3[x^2 - (5y)^2]$
$= 3(x + 5y)(x - 5y)$

The difference of two squares can be used to evaluate some expressions that involve numbers in a faster way.

Example 9.13

Evaluate:

(a) $99^2 - 1$ (b) $8.49^2 - 1.51^2$

(c) 1012×988

Solutions

(a) $99^2 - 1 = 99^2 - 1^2$
$= (99 - 1)(99 + 1)$
$= 98 \times 100$
$= 9800$

(b) $8.49^2 - 1.51^2 = (8.49 - 1.51)(8.49 + 1.51)$
$= 6.98 \times 10$
$= 69.8$

(c) $1,012 \times 988 = (1,000 + 12)(1,000 - 12)$
$= 1,000^2 - 12^2$
$= 1,000,000 - 144$
$= 999,856$

Exercise 9.5

1. Factorize:

(a) $x^2 - 2^2$ (b) $x^2 - 5^2$

(c) $7^2 - n^2$ (d) $1 - t^2$

(e) $(3a)^2 - 1$ (f) $4c^2 - 25^2$

(g) $25 - 49r^2$ (h) $p^2 - 36q^2$

(i) $p^2q^2 - 1$ (j) $48a^2 - 3t^2$

(k) $72m^2 - 2$ (l) $4p^2 - 4q^2$

(m) $x^4 - 1$ (n) $a^3b - ab^3$

(o) $nu^2 - nv^2$

2. Calculate:

(a) $78.3^2 - 21.7^2$ (b) $923^2 - 77^2$

(c) $0.843^2 - 0.157^2$ (d) $75^2 - 25^2$

(e) 94×106 (f) $975 \times 1,025$

(g) $3 \times 25^2 - 3 \times 75^2$ (h) $5 \times 12^2 - 5 \times 88^2$

(i) $6 \times 53^2 - 6 \times 47^2$ (j) $\frac{22}{7} \times 56^2 - \frac{22}{7} \times 56^2$

3. Factorize completely

(a) $x^2y^2 - 16$ (b) $1 - p^2q^2$

(c) $9x^2 - y^2$ (d) $a^2b^2 - c^2$

(e) $3x^2 - 27$ (f) $2a^2 - 8$

(g) $5a^2b^2 - 45c^2$ (h) $8a^2 - 32b^2c^2$

(i) $100 - x^2$ (j) $16x^2 - 25$

(k) $81 - x^2y^2$ (l) $x^2y^2 - p^2q^2$

(m) $3x^2 - 48y^2z^2$ (n) $5b^2c^2 - 20d^2g^2$

Solving quadratic equations

We solve an equation by finding the value of the unknown. For any two numbers, p and q, if pq = 0, then, either:

(a) p = 0 which means $0 \times q = 0$
(b) q = 0 which means $p \times 0 = 0$
(c) p = q = 0 which means $0 \times 0 = 0$

Similarly, if (x + 2)(x+ 3) = 0, then either x + 2 = 0 or x + 3 = 0.
Solving for x, gives either x = -2 or x = -3.
Thus, in order to solve a quadratic equation, the quadratic expression is factorized so that the equation is in the form (x + a)(x + b) = 0.

Example 9.14

Solve (x - 4)(x + 1) = 0

Solution

If $(x - 4)(x + 1) = 0$, then either $x - 4 = 0$ or $x + 1 = 0$.
Therefore, $x = 4$ or $x = -1$.
Hence the roots of the equation $(x - 4)(x + 1) = 0$ are 4 and -1.

Example 9.15

Factorize $x^2 + 7x + 6 = 0$

Solution

$x^2 + 7x + 6 = 0$
Factorizing, $x^2 + 7x + 6$, gives $(x + 6)(x + 1) = 0$.
Therefore, $x + 6 = 0$ or $x + 1 = 0$; which means $x = -6$ or $x = -1$

These are the only two values of x which satisfy the equation $x^2 + 7x + 6 = 0$. We can check these solutions by substituting each of them in the equation.
Thus, when $x = -6$,

$(-6)^2 + 7(-6) + 6 = 36 - 42 + 6 = 0$

And when $x = -1$

$(-1)^2 + 7(-1) + 6 = 1 - 7 + 6 = 0$

Note: *Every quadratic equation has two solutions*.

Example 9.16

Solve: $x^2 + x - 72 = 0$

Solution

$x^2 + x - 72 = 0$
$\Leftrightarrow (x - 8)(x + 9) = 0. \quad \Rightarrow x - 8 = 0$ or $x + 9 = 0$
$\therefore \quad x = 8$ or $x = -9$.

Example 9.17

Solve: $x^2 - x - 29 = 1$

Solution

Always ensure that the quadratic expression is equated to zero. This is the only time the method used in the examples above can apply.
Thus, $x^2 - x - 29 = 1$ should be rewritten as $x^2 - x - 29 - 1 = 0$. That is, $x^2 - x - 30 = 0$.

The factors of 30, whose sum is 1, are -6 and 5.
Therefore, $(x - 6)(x + 5) = 0$.
Either $x - 6 = 0$ or $x + 5 = 0$
$\therefore x = 6$ or $x = -5$
The roots are -5 and 6.

Example 9.18

Solve:
(a) $x^2 - 49 = 0$
(b) $x^2 - 6x = 0$
(c) $x^2 - 16x + 64 = 0$
(d) $6x^2 + 5x - 4 = 0$

Solutions

(a) $x^2 - 49 = 0$ can be written as $x^2 - 7^2 = 0$
$(x - 7)(x + 7) = 0$
$x - 7 = 0$ or $x + 7 = 0 \Rightarrow x = 7$ or $x = -7$.
The roots are -7 and 7.

(b) Factorizing $x^2 - 6x = 0$ gives $x(x - 6) = 0$
Either $x = 0$ or $x - 6 = 0$
$\Rightarrow x = 0$ or $x = 6$
The roots are 0 and 6.

(c) $x^2 - 16x + 64 = 0$
Factorizing $x^2 - 16x + 64 = 0$ gives $(x - 8)(x - 8) = 0$
Either $x - 8 = 0$ or $x - 8 = 0$
$\Rightarrow x = 8$ or $x = 8$
The roots are 8 and 8.

Note: *$x^2 - 16x + 64$ is a perfect square and therefore it has identical factors. The equation $x^2 - 16x + 64 = 0$ has two equal roots.*

(d) $6x^2 + 5x - 4 = 0$
When the coefficient of x^2,(in this case it is 6), in the quadratic expression is numerically greater than 1, we proceed as follows when factorizing:
- Multiply the coefficient of x^2 by the constant term, i.e. $6 \times -4 = -24$.
- Find the factors of -24 whose sum is 5, (the coefficient of x), i.e. -3 and 8.
- Rewrite the equation as:

$6x^2 + (-3 + 8)x - 4 = 0$
$\Rightarrow 6x^2 - 3x + 8x - 4 = 0$
$\Rightarrow 3x(2x - 1) + 4(2x - 1) = 0$
$\Rightarrow (2x - 1)(3x + 4) = 0$
Then, either $2x - 1 = 0$ or $3x + 4 = 0$
$\Rightarrow \quad 2x = 1$ or $3x = -4$
$\therefore \quad x = \frac{1}{2}$ or $x = -\frac{4}{3}$.

Example 9.19

Factorize $3x^2 - 22x + 7$. Hence solve $3x^2 - 22x + 7 = 0$

Solution

$3x^2 - 22x + 7$

Multiplying 3 by 7, gives 21. The factors of 21 whose sum is -22, are -1 and -21.
Then, $3x^2 - 22x + 7 \Leftrightarrow 3x^2 + [-1 + (-21)]x + 7$
$\Leftrightarrow 3x^2 - 1x - 21x + 7$
$\Leftrightarrow x(3x - 1) -7(3x - 1)$
$\Leftrightarrow (3x - 1)(x - 7)$
Hence, $3x^2 - 22x + 7 = (3x - 1)(x - 7)$.

The equation $3x^2 - 22x + 7 = 0$ can be written as $(3x - 1)(x - 7) = 0$
Either $3x - 1 = 0$ or $x - 7 = 0$
$3x = 1$ or $x = 7$

∴ $x = \frac{1}{3}$ or $x = 7$

Exercise 9.6: Solve.

1. $(x + 4)(x + 2) = 0$
2. $(x - 5)(x - 3) = 0$
3. $(x - 5)(x + 7) = 0$
4. $(x - 9)^2 = 0$
5. $(x + 13)^2 = 0$
6. $x^2 + x - 12 = 0$
7. $x^2 + 9x + 14 = 0$
8. $x^2 - 11x - 12 = 0$
9. $u^2 + 6u + 9 = 0$
10. $t^2 - t - 42 = 0$
11. $a^2 - 2a + 1 = 0$
12. $y^2 + 8y = 0$
13. $x^2 + 10x = 24$
14. $x^2 = 4x - 3$
15. $v^2 - 36 = 0$
16. $4x^2 - 9 = 0$
17. $1 - y^2 = 0$
18. $81 - 16q^2 = 0$
19. $x^2 = 6x$
20. $5x^2 = 45$
21. $6x(2x + 3) = 0$
22. $-4x(3x - 5) = 0$
23. $(3x - 2)(2x + 1) = 0$
24. $6x^2 - 29x + 35 = 0$
25. $6x^2 - x + 1 = 0$
26. $(4x - 3)(3x - 4) = 0$
27. $4y^2 - 48 = 16$
28. $3y^2 - 24 = 3$

Chapter 10

SIMILARITIES AND ENLARGEMENTS

SIMILARITIES

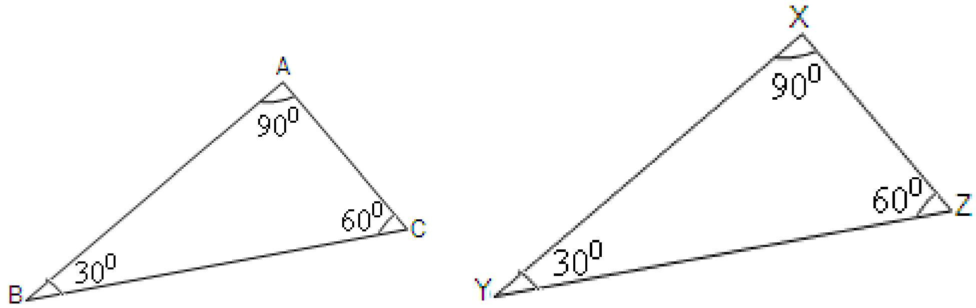
Geometrical figures are said to be similar if they are the same shape but different in size. Everyday we see similar shapes. The following are some examples:

- A toy car and a real car,
- A photograph and the person,
- Drawn plans of houses to represent actual houses.

Properties of similar figures

Similar triangles

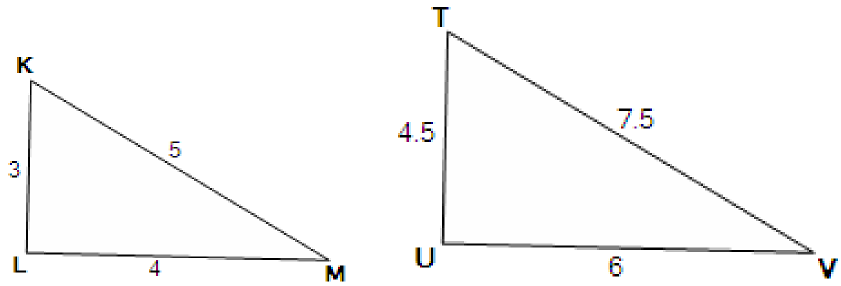
The figure below shows two similar triangles ABC and XYZ.



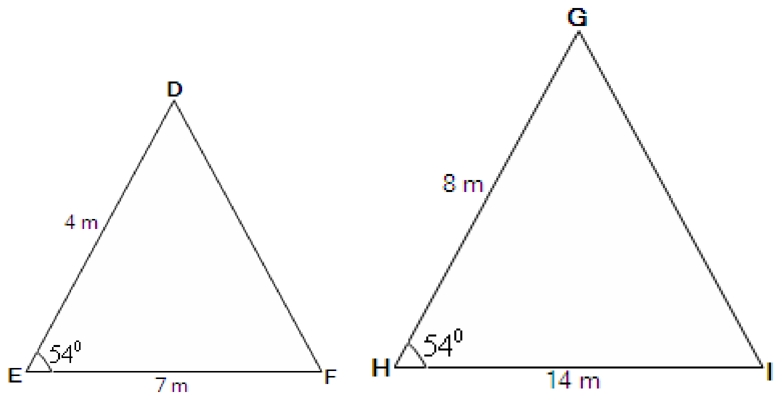
Triangles that have corresponding angles equal are said to be equiangular.
Two triangles are similar if:

(a) the angles of one triangle are equal to the corresponding angles of the other triangle, for example, triangle ABC is similar to triangle XYZ because $\angle BAC = \angle YXZ$, $\angle ABC = \angle XYZ$ and $\angle ACB = \angle XZY$.

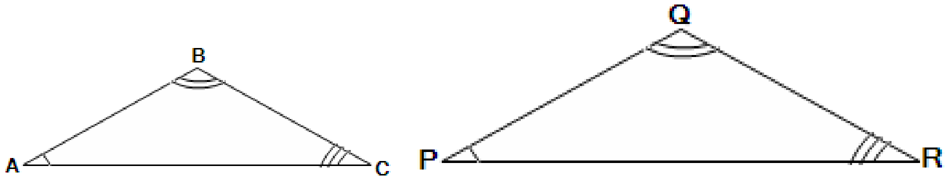
(b) The corresponding sides are all in the same ratio, for example, triangle KLM is similar to triangle TUV because $\frac{KL}{TU} = \frac{LM}{UV} = \frac{KM}{TV} = \frac{2}{3}$



(c) there is one pair of equal angles and the two sides including these equal angles are in the same ratio. For example, triangle DEF is similar to triangle GHI because $\angle DEF = \angle GHI$ and $\frac{DE}{GH} = \frac{EF}{HI} = \frac{1}{2}$.



Similar triangles are named in the same order of corresponding vertices. For example, in the figure below, triangle ABC is similar to triangle PQR.

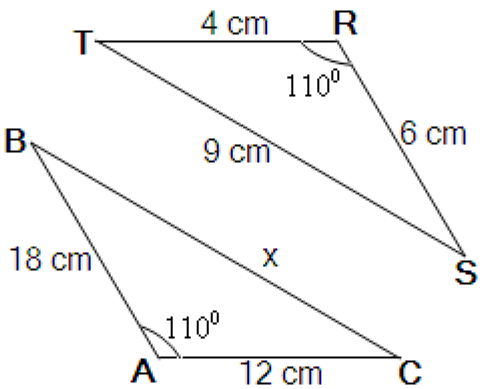


This means that $\angle BAC = \angle QPR$, $\angle ABC = \angle PQR$, $\angle BCA = \angle QRP$. And $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$ or $\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC}$.

Note: When comparing the sides of two similar triangles, the sides in the numerator belong to one triangle and the sides in the denominator belong to the other triangle throughout.

Example 10.1

Show that triangles ABC and RST in the figure below are similar and find the value of x.



Solution

The pair of side including the given angles are AB and AC in ΔABC and RS and RT in ΔRST.

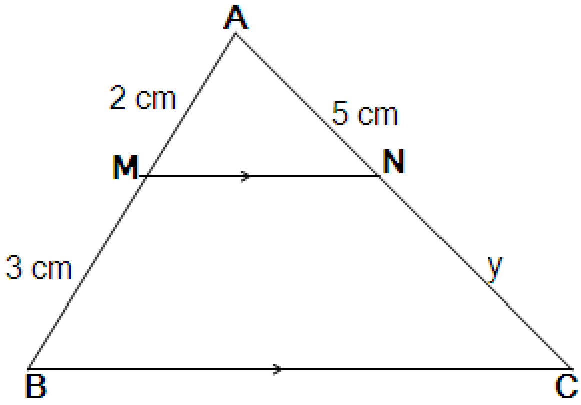
$\frac{AB}{RS} = \frac{18}{6} = 3$ and $\frac{AC}{RT} = \frac{12}{4} = 3$.

Since the pair of sides including the equal angles are in the same ratio, ΔABC and ΔRST are similar. Therefore, $\frac{BC}{TS} = \frac{AB}{RS}$ that is $\frac{x}{9} = \frac{18}{6}$.

$$x = \frac{18 \times 9}{6} = 3 \times 9 = 27 \text{ cm}.$$

Example 10.2

In triangle ABC, M is a point on line AB and N is a point on line AC such that, AM = 2 cm, MB = 3 cm and AN = 5 cm.



Given that MN is parallel to BC,

(a) show that ΔAMN and ΔABC are similar,

(b) calculate the value of y.

Solutions

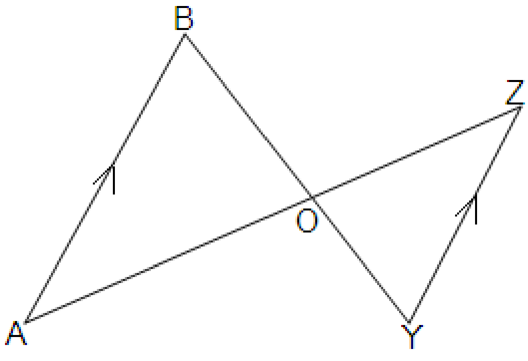
$\angle$AMN = $\angle$ABC (corresponding angles).
$\angle$ANM = $\angle$ ACB (corresponding angles).
$\angle$MAN is the same as $\angle$BAC since it is shared in both ΔABC and ΔAMN. Therefore, ΔABC and ΔAMN are equiangular and similar.

$\frac{AM}{AB}=\frac{AN}{AC}$. That is, $\frac{2}{5}=\frac{5}{5+y}$

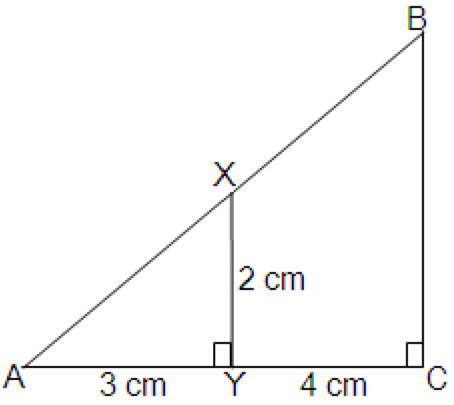
$$2(5 + y) = 25$$
$$5 + y = 12.5$$
$$y = 12.5 - 5 = 7.5 \text{ cm.}$$

Exercise 10.1

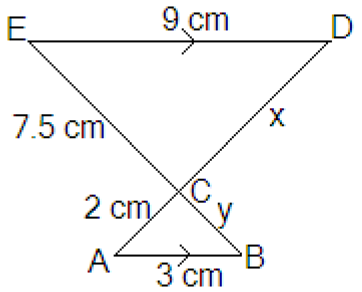
1. In the figure below, AB is parallel to YZ. Show that ΔABO is similar to ΔOYZ.



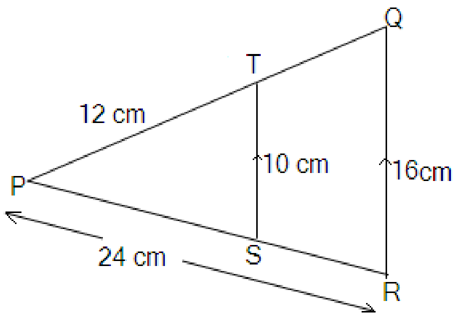
2. Find the length of BC



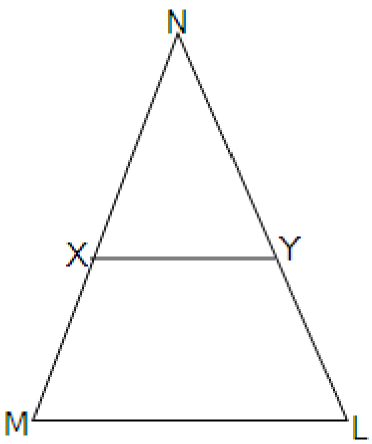
3. Find the values of x and y.



4. Find the lengths of lines TQ and PS.



5. In ΔLMN, MN = NL, X is the mid-point of MN and Y is the id-point of NL.

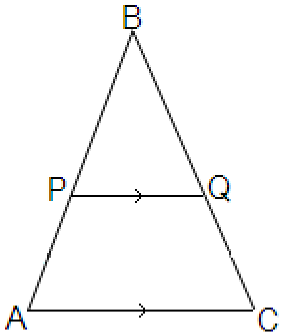


(a) write down the values of $\frac{NY}{NL}$ and $\frac{NX}{NM}$.

(b) Are ΔYXN and ΔLMN similar?

(c) Is XY parallel to ML?

6. In ΔABC, PQ is parallel to AC. AB = 6 cm, BC = 10 cm and BQ = 4 cm.



(a) Find BP

(b) Write down the values of BP:PA and BQ:QC.

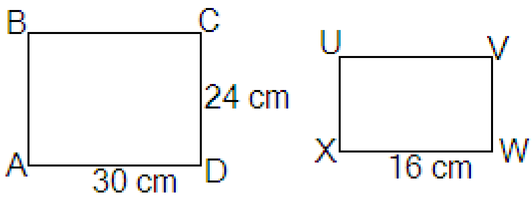
Other similar figures

Rectangles are similar if their lengths and breadths are in the same ratio. It is important to remember that angles of a rectangle are all right angles.

Exercise 10.2

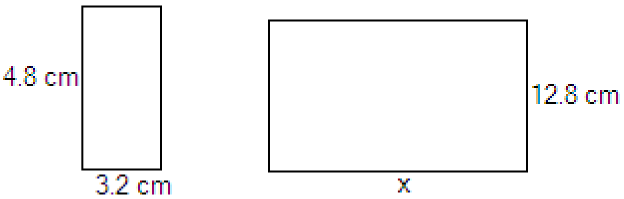
1. The pairs of figures are similar in each of the following cases.

(a)



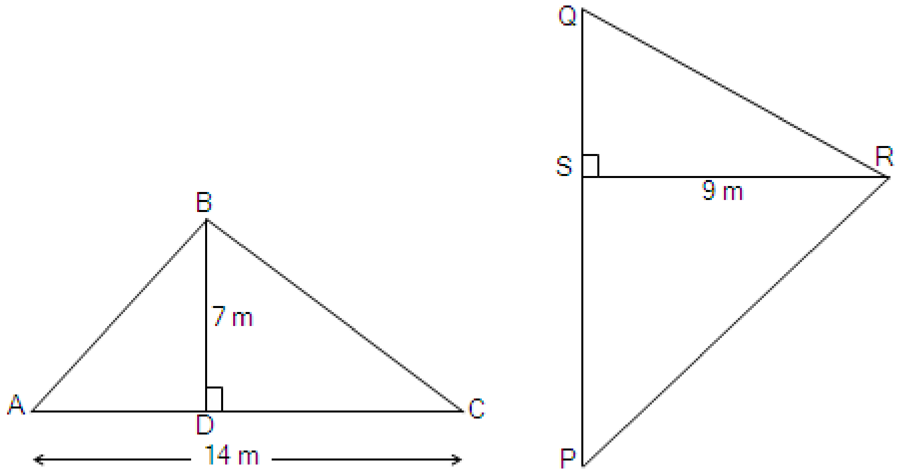
Find VW.

(b)



Find the value of x.

(c)



Find PQ.

2. A packet of cooking oil is in the shape of a cuboid 10 cm long, 8 cm wide and 4 cm high. The packets of oil are packed in a similar box 24 cm high. Calculate the length and breadth of the box.

3. A cake is in the shape of a cuboid 10 cm long, 6 cm wide and 4 cm high. The cake is packed in a similar box 24 cm wide. Calculate the length and the height of the box.

4. A model of a house is 28 cm long and 20 cm wide. If the length of the house is 42 m what is its width?

5. A cylindrical tin has a diameter 36 cm. It is similar to a cylindrical tank with a diameter 2m and a height 4.5 m. Calculate the height of the tin.

6. Mary's height in a photograph is 5 cm. What is her actual height if the ratio of her photograph to her actual height is 1 : 32?

7. A toy car, constructed to a scale of 1 : 140 is 1.6 cm long. What is the length of the actual car?

ENLARGEMENT

An enlargement is a transformation which **increases** or **decreases** the size of an object, in a given ratio. Under an enlargement:

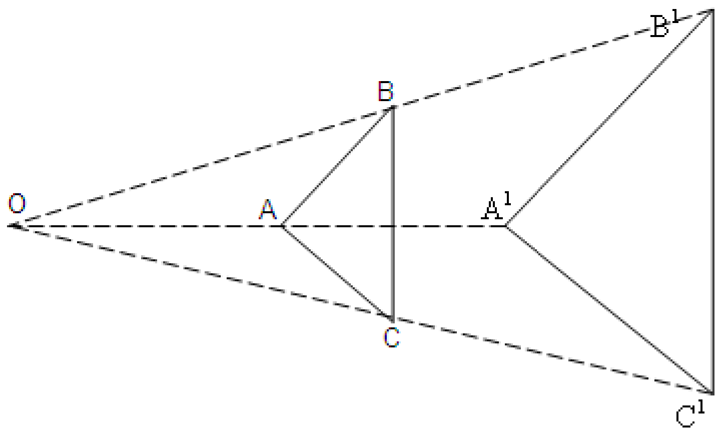
(a) the object lines and their images are parallel,
(b) the lengths of the lines of the objects and their images are in the same ratio,
(c) the angles remain the same.

Under an enlargement, the object and its image are similar.

Constructing similar figures

Positive scale factor

The figure below shows triangle ABC and its image $A^1B^1C^1$ under an enlargement transformation.



Line AA^1, BB^1 and CC^1 meet at O. The point, O, where these lines meet is called the **centre of enlargement.**

Note: ΔOAB and ΔOA^1B^1 are similar.

What other triangles are similar? What can you say about the ratios

$\frac{OA^1}{OA}, \frac{OB^1}{OB}, \frac{OC^1}{OC}$?

The ratio $\frac{OA^1}{OA}$ is called the **linear scale factor** because it is the ratio of the lengths of the line segments.

What other ratio would give the same scale factor?

The scale factor of enlargement is always positive whenever the object and its image are on the same side of the centre of enlargement.

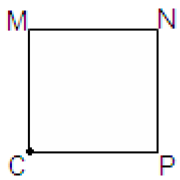
Note:

- When the scale factor is greater than 1, then the object is smaller than the image and it lies between the image and the centre of enlargement.
- When the scale factor is positive and less than 1, then the image is smaller than the object and it lies between the object and the centre of enlargement.
- The scale factor is a ratio and has no units.

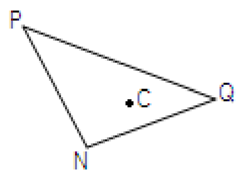
Example 10.3

Using point C as the centre of enlargement, enlarge the following figures with a scale factor of 2.

(a)

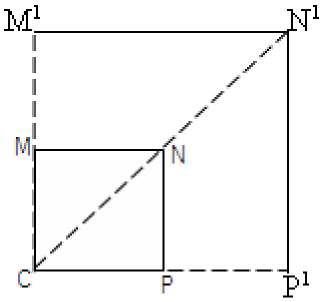


(b)

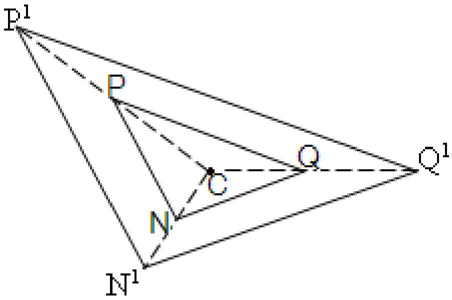


Solutions

(a) The image of each vertex will be twice as far away from the centre as the object is. The distance CM^1 is twice the distance CM. Draw a line from C to M and extend it to M^1, a distance 2CM. Repeat for N^1 and P^1 to get the figure below.

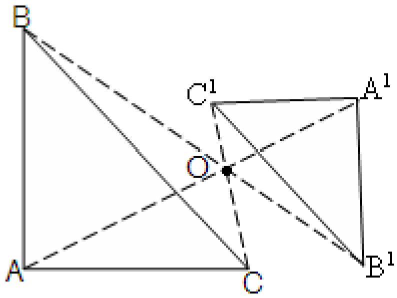


(b) Measure line CQ. Extend line CQ to a point Q^1 such that $2CQ = CQ^1$. Repeat the same with points P and N to get the figure below.

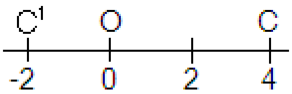


Negative scale factor

Consider the following figure in which triangle ABC is taken as the object and $\Delta A^1B^1C^1$ as its image under an enlargement about centre O.



The object ΔABC and the image $\Delta A^1B^1C^1$ are on the opposite sides of the centre of enlargement, O.
Suppose OC = 4 cm and OC^1 = 2 cm. Marking line C^1OC on a number line with O at zero and C at 4 units from O, then C^1 is -2 units. See the figure below.



The scale factor = $\frac{\text{image distance from O}}{\text{object distance from O}}$.

Thus, $\frac{OC^1}{OC} = \frac{-2}{4} = -\frac{1}{2}$ or $OC^1 = -\frac{1}{2}$ OC.

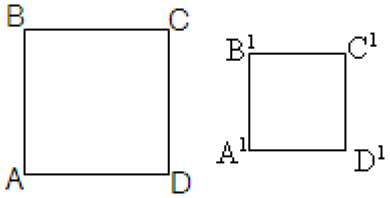
Therefore, the scale factor is $-\frac{1}{2}$.

Locating the centre of enlargement

In order to locate the centre of enlargement when given an object and its image, join at least two points on the object to their corresponding points on the image with straight lines. The two straight lines intersect at the centre of enlargement.

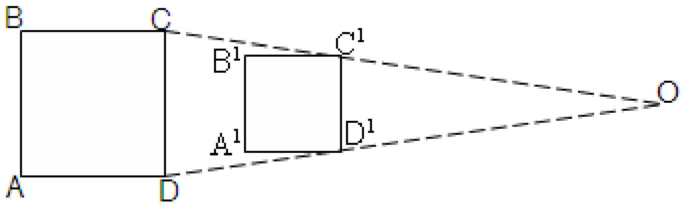
Example 10.4

Find the centre of enlargement of the squares given below.



Solution

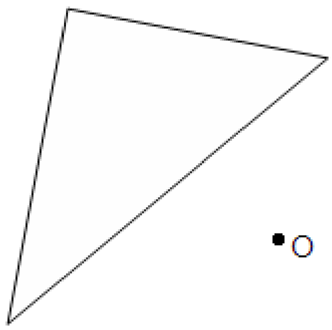
We join at least two points on the object, for example, C and D to their corresponding points on the image, that is, C^1 and D^1 as shown below.
Therefore, the centre of enlargement is O.



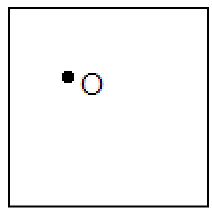
Exercise 10.3

1. Copy each of the following figures. Using point O as the centre of enlargement, enlarge it with a scale factor of 2.

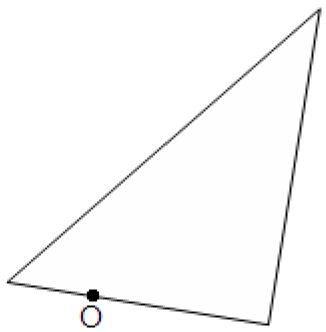
(a)



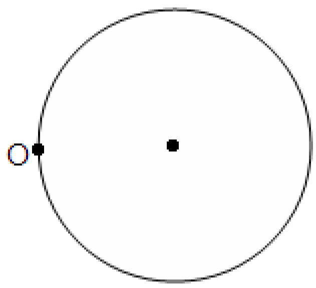
(b)



(c)



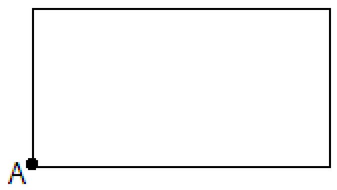
(d)



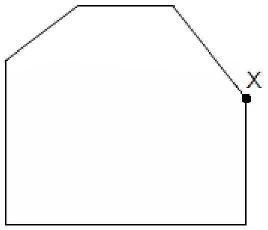
2. Repeat question 1 using a scale factor of -2.

3. Copy the following figures and make enlargements at the stated point and scale factor given.

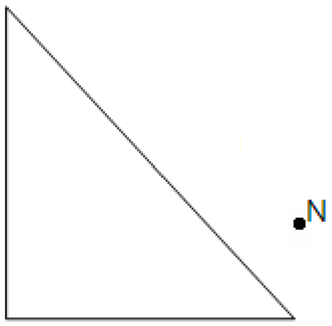
(a) [A, 3]



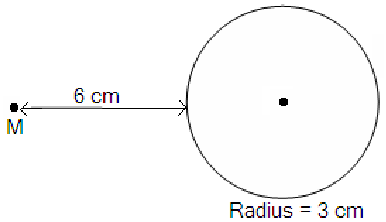
(b) [X, $1\frac{1}{2}$]



(c) [N, $-\frac{1}{2}$]

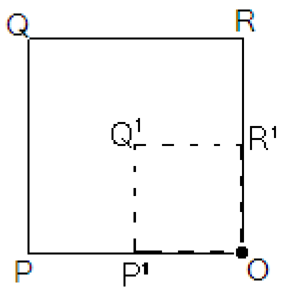


(d) $\left|M, \frac{1}{3}\right|$

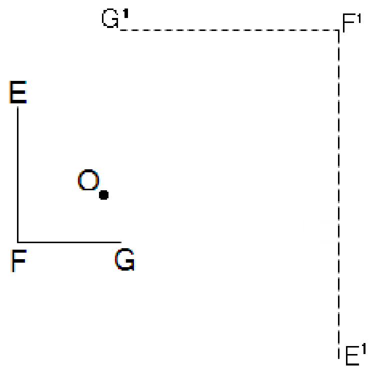


4. In each of the following pairs of figures, the figure in broken lines is the image. Using the centre of enlargement, O, state the scale factor.

(a)

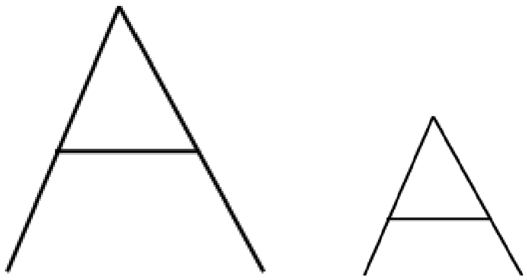


(b)

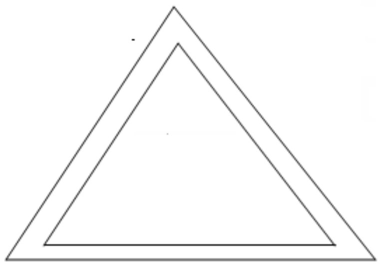


5. Trace the following figures and in each case locate the centre of enlargement.

(a)

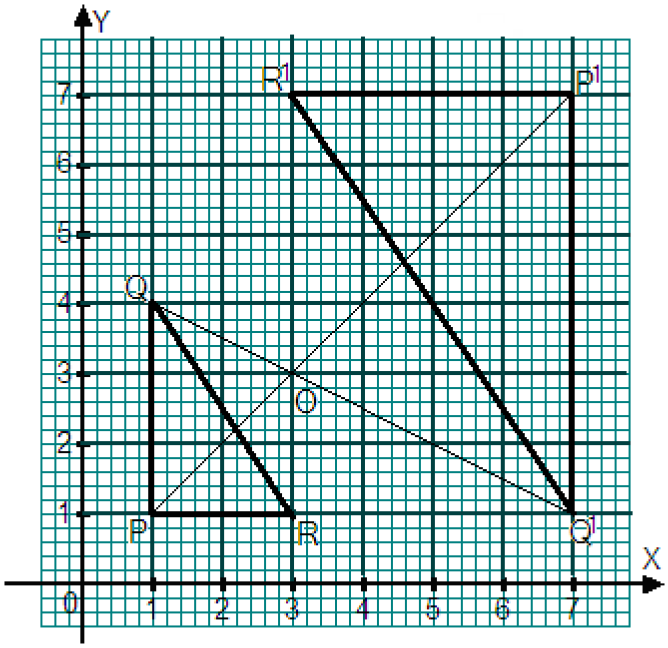


(b)



Enlargement on the Cartesian plane

Triangle PQR, with vertices P(1, 1), Q(1, 4) and R(3, 1), undergoes an enlargement, centre (3, 3), with a scale factor of -2. Find the coordinates of the vertices of P^1, Q^1 and R^1. From the diagram, $R^1(3, 7)$, $Q^1(7, 1)$ and $P^1(7, 7)$.



Note: *All the lines joining each vertex on the object to its image pass through O, the centre of enlargement.*

Exercise 10.4

1. On graph paper, plot the points A(1, 3), B(3, 3) and C(3, 1). Join the points to form ΔABC. Find the coordinates of A^1, B^1 and C^1 after an enlargement of:
   (a) scale factor 2 with centre (0, 0).
   (b) scale factor -1 with centre (0, 0).
   (c) scale factor 3 with centre (2, 2).
   (d) scale factor -4 with centre (2, 2).

2. Triangle PQR is such that P(0, 0), Q(0, 2) and R(1, 0). It is enlarged to form triangle $P^1(2, 2)$, $Q^1(2, 8)$ and $R^1(5, 2)$. Locate the centre of enlargement and state the scale factor.

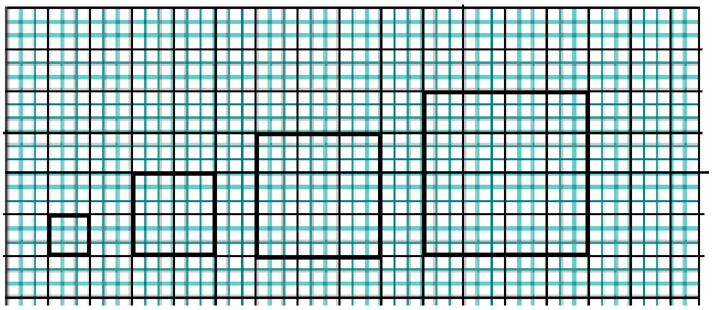
3. Points $A_1(-3, 0)$ and $B_1(-1, 0)$ are the images of A(6, 6) and B(2, 6) under an enlargement. Locate the centre of enlargement and the scale factor.

4. A square with vertices A(-2, -2), B(-2, 2), C(2, 2) and D(2, -2), undergoes an enlargement with the origin as centre and a scale factor of k. Find the coordinates of the vertices of the image when k is:
   (a) $\frac{1}{2}$ (b) -1
   (c) $2\frac{1}{2}$

5. A semicircle has a diameter MN. The coordinates of points M and N are M(1, 4) and N(5, 0). The semicircle is enlarged about point (3, 2) by a scale factor of -1. Construct both the object and its image on the same pair of axes. State the coordinates of M^1 and N^1.

Areas of similar figures

The four squares in the figure below are similar. The smallest square has a Side of 1 unit length and an area of one square unit. The ratio of the lengths of their sides is 1:2:3:4. Counting the number of squares in each figure gives the ratio of their areas as 1:4:9:16.
We know that $1:4:9:16 = 1^2:2^2:3^2:4^2$. Thus **the ratio of the areas is equal to the ratio of the squares of the corresponding lengths**.
In general, if two figures are similar and their corresponding lengths are in the ratio a:b then their areas are in the ratio $a^2:b^2$.



Example 10.5

Two similar triangles have their corresponding lengths in the ratio 2:3.
(a) Find the ratio of their areas.

(b) Given that the smaller triangle has an area of 20 cm^2, calculate the area of the larger triangle.

Solution

(a) Ratio of the corresponding lengths = 2:3
Ratio of the corresponding areas = 4:9

(b) Let the area of the larger triangle be x

Then, $\frac{x}{20} = \frac{9}{4}$

$x = \frac{9}{4} \times 20 = 45$

The area of the larger triangle is 45 cm^2.

Example 10.6

Triangles ABC and DEF are similar. If the area of ABC is 9.8 cm^2 and the area of ΔDEF is 5.0 cm^2, find the value of AB:DE.
Solution

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{9.8}{5.0} = \frac{98}{50} = \frac{49}{25}.$$

But $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2}$. Therefore, $\frac{AB^2}{DE^2} = \frac{49}{25}$

$$\sqrt{\frac{AB^2}{DE^2}} = \frac{AB}{DE} = \sqrt{\frac{49}{25}} = \frac{7}{5}$$

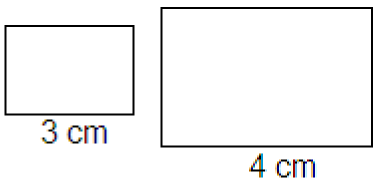
AB:DE = 7:5

Note: *We obtain the linear ratio by finding the square root of the area ratio. It is important to simplify the area ratio to its lowest terms before finding the square root to avoid tedious calculations.*

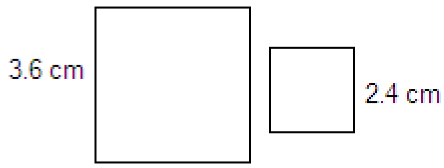
Exercise 10.5

1. For each pair of similar figures below, write down the ratios of their areas.

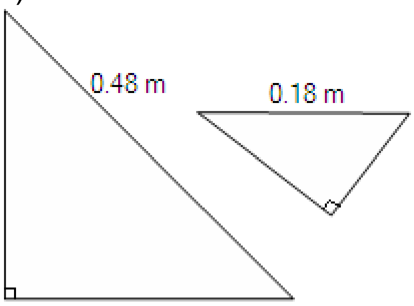
(a)



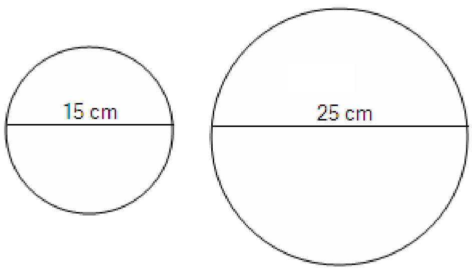
(b)



(c)

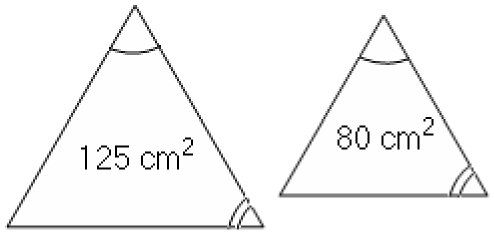


(d)

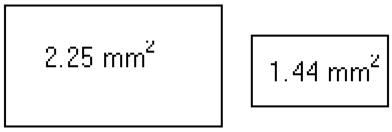


2. Find the linear ratios for each of the following pairs of figures:

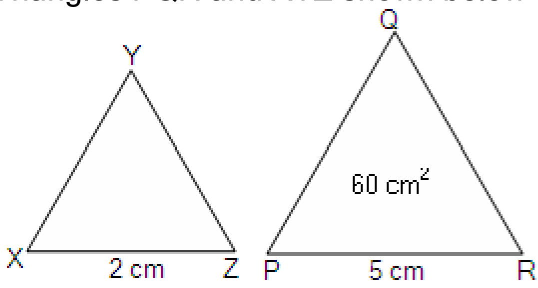
(a)



(b)



3. Triangles PQR and XYZ shown below are similar. Find the area of ΔXYZ



4. The length of a model car is 12 cm. If the actual length of the car is 2.16 m:
   (a) what is their linear scale factor?
   (b) what is the area occupied by the actual car if the model car occupies an area of 72 cm^2?

5. A plan of a building is to the scale 1:50. What is the actual area of the room that is 2 cm^2 on the plan?

6. The radii of two balls are in the ratio 2:3. Calculate the surface area of the smaller ball if the bigger ball has a surface area of 28.8 cm^2.

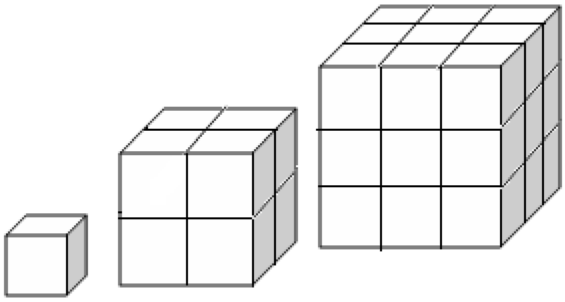
7. The heights of two similar cylinders are in the ratio 1:4. What is the:
   (a) ratio of their base radii?
   (b) surface area of the larger cylinder if the smaller cylinder has a surface area of 12.5 cm^2?

8. A picture is reduced by a scale of 1:12. If the original picture measured 48 cm by 72 cm, what is the area of the new picture?

9. Given that the ratio of the areas of two similar solids is 9:25,
   (a) what is the linear scale factor between the two solids?
   (b) what is the length of the bigger solid, if the length of the smaller solid is 1.11 m.

Volume of similar figures

The three cubes in the figure below are similar.



The smallest cube has all its edges as one unit length. Therefore, it has a volume of one cubic unit. The ratio of the lengths of their edges is 1:2:3. Counting the number of cubes in each figure gives the ratio of their volumes as $1:8:27 = 1^3:2^3:3^3$.

The ratio of the volumes of similar solids is equal to the ratio of the cubes of the corresponding lengths.

In general, if two solids are similar and their corresponding lengths are in the ratio a:b, then their volumes are in the ratio $a^3:b^3$.

Example 10.7

Two similar cylinders have their lengths in the ratio 3:5.

(a) Find the ratio of their volumes.

(b) Given that the bigger cylinder has a volume of 750 cm^3, calculate the volume of the smallest cylinder.

Solutions

(a) The ratio of the corresponding lengths is 3:5. Therefore, the ratio of the volumes is $3^3:5^3$. = 27:125.

(b) Let the volume of the smaller cylinder be v. Then,

$\frac{v}{750} = \frac{27}{125}$. This means,

$v = \frac{27}{125} \times 750 = 162 \text{ cm}^3$.

Example 10.8

The volume of two similar cones are in the ratio 1:27.

(a) Calculate the volume of the smaller cone given that the volume of the larger cone is 64 cm^3.

(b) What is the ratio of their base areas?

Solution

(a) $\frac{\text{Volume of the smaller cone}}{\text{Volume of thelarger cone}} = \frac{1}{27}$. This means,

$\frac{\text{Volume of the smaller cone}}{64} = \frac{1}{27}$

Therefore, the volume of the smaller cone = $\frac{1}{27} \times 64 = \frac{64}{27} \text{ cm}^3 = 2.37 \text{ cm}^3$

(b) Volume ratio = 1:27
Linear ratio = 1:3
Area ratio = $1^2:3^2$
= 1:9

Example 10.9

The radius of a spherical balloon increases by 4%. Find the percentage increase in its:

(a) surface area, (b) volume.

Solution

(a) let the original radius be r. The new radius is given by

$\frac{104}{100} \times r = 1.04r$

Therefore, $\frac{\text{new radius}}{\text{old radius}} = \frac{1.04r}{r} = \frac{1.04}{1}$

$\frac{\text{New surface area}}{\text{Original surface acrea}} = \frac{(1.04)^2}{1^2} = 1.0816$

Therefore, the new surface area = 1.0816 of the original surface area. Expressing the new surface area as a percentage of the original surface area we get 108.16%. The surface area has increased by 8.16%.

(b) $\frac{\text{New Volume}}{\text{Original Volume}} = \frac{(1.04)^2}{1^2} = 1.1249$ (to 4 d.p)

New volume = 1.1249 of the original volume which gives the percentage of the new volume as 112.49%. Therefore, the volume has increased by 12.49%.

Note: *We did not need to know the original radius of the balloon.*

Exercise 10.6

1. The radii of two similar cylinders are in the ratio 1:4. What is the ratio of their volumes?

2. The heights of two similar cuboids are in the ratio 3:5. Find the ratio of their capacities.

3. The volumes of two similar jugs are in the ratio 8:125. What is the ratio of their heights?

4. The volumes of two similar pyramids are in the ratio 216:27. What is the ratio of their:
   (a) heights?
   (b) base diagonals?

5. A soap bubble has a radius of 2 cm. If its radius increases by 1 cm, find by how much does its volume increase.

6. The heights of three cylinders are 3 cm, 9 cm and 15 cm. If the larger cylinder has a capacity of 725 milliliters, calculate the capacities of the other cylinders.

7. All the linear dimensions of a model house are $\frac{1}{50}$ of the actual

dimensions of the house. Find the ratio of:

(a) the area of the kitchen,

(b) volume of the space occupied by the garage.

8. Two similar bottles are such that one is thrice as high as the other. What are the ratios of their:

(a) surface area?

(b) volumes?

(c) number of bottle tops

9. Two similar jars hold $21\frac{1}{3}$ litres and 9 litres respectively.

(a) Find the ratio of their radii.

(b) If the smaller jar has a radius of 12 cm, find the radius of the larger jar.

10. The height of a rectangular water tank is 2.4 m. When full it contains 8,000 litres. How many litres would fill a similar tank that is 9.6 m high?

11. The top surface areas of water in two similar swimming pools are 140 m^2 and 210 m^2. What volume of water does the smaller pool contain if the larger one has 54,000 litres of water?

12. A statue of volume 30,720 cm^3 is represented by a model of volume 480 cm^3.

(a) What is the ratio of their lengths?

(b) What is the ratio of their surface areas?

(c) If it costs sh. 1500 to paint the model, what would be the cost of painting the statue?

13. The radius of a spherical soap bubble increases by 20%. Find, correct to the nearest whole number, the percentage increase in its:

(a) surface area,

(b) volume.

14. The surface area of a cone increases by 300%. Find the percentage increase in its:

(a) slant height,

(b) volume.

Applications of scale factors

Example 10.10

In a plan of a house, the scale of the length is shown as 1:50. Calculate:

(a) the length of a room which is 8.5 cm on the plan,

(b) the height of a door on the plan if the real height is 2 m.

Solutions

The scale of 1:50 means 1 cm on the plan represents 50 cm on the ground.

(a) If 1 cm represents 50 cm, then 8.5 cm represents 8.5 × 50 = 425 cm or 4.25 m.

(b) 50 cm is represented by 1 cm on the plan. Therefore, 200 cm is represented by $\frac{1 \times 200}{50} = 4\text{cm}$. The height of the window on the plan is 4 cm.

Example 10.11

The scale of a map is 1:5,000. On the map, the area representing a building is 2 cm^2. What is the actual area, in square metres, occupied by the building?

Solution

Linear scale is 1:5,000

Area scale = $1^2:5,000^2$

$= 1: 25,000,000$ or $\frac{1}{25000000}$

$$\frac{\text{Actual area}}{2\,cm^2} = \frac{25000000}{1}$$

Which means, actual area = 25,000,000 × 2 = 50,000,000 cm^2

But, 10.000 cm^2 = 1 m^2. Therefore, the area occupied by the building is 5,000 m^2.

Exercise 10.7

1. A girl whose eye is 1.5 m above the ground can just see over the top of a wall 2.4 m high and the top of a tower which is 450 m beyond the wall. If the girl is 22.5 m from the wall, what is the height of the tower?

2. Two similar vases have their heights in the ratio 3:2. What is the ratio of:
   (a) their surface areas?
   (b) their volumes?

3. The size of a picture is reduced so that it is one-tenth of the original size. What was the original width, if the reduced width is 6.8 cm?

4. A plan of a building is to a scale of 1:50.What is the area of a room which is 8 cm^2 on the plan?

5. A map is drawn to a scale of 1:100,000. What is the area on the map of an estate that is 80,000 m^2 on the ground?

6. The linear dimensions of a model car are $\frac{1}{16}$ of the dimensions of the actual car.
   (a) What is the area of the windscreen of the actual car, if the windscreen of the model car has an area of 3 cm^2?
   (b) If the capacity of the boot of the model car is 12cm^3, find the capacity of the boot of the actual car.

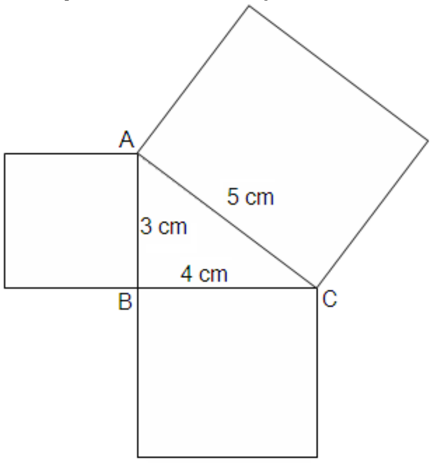
7. In a photograph, a boy has a ball which is 15 times smaller than the actual ball.
   (a) If the diameter of the ball in the photograph is 1.2 cm, what is the diameter of the actual ball?
   (b) What is the boys height in the photograph if his actual height is 150 cm?

Chapter 11

THE PYTHAGORAS' THEOREM

Deriving the Pythagoras' theorem

Draw a right angled triangle ABC in which $\angle ABC = 90^0$, AB = 3 cm and BC = 4 cm. Measure the length of the hypotenuse. On sides AB, BC and CA respectively, construct squares as shown in the figure below.

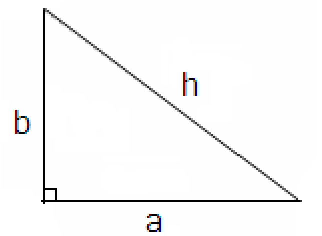


Find the area of each of these squares.
From the figure, you will find that:

- The area of the square constructed on side AB = 3 cm × 3 cm = 9 cm^2.
- The area of the square constructed on side BC = 4 cm × 4 cm = 16 cm^2.
- The area of the square constructed on side AC = 5 cm × 5 cm = 25 cm^2.

The sum of the areas of the squares drawn on sides AB and BC = (9 + 16) = 25 cm^2.
Therefore, the sum of the areas of the squares on sides AB and BC is equal to the area of the square on side AC. Thus, $AC^2 = AB^2 + BC^2$.
This relationship is known as **Pythagoras' theorem**.
In general, if the length of the hypotenuse in a right angled triangle is h and the other two sides are a and b respectively, then

$$h^2 = a^2 + b^2.$$

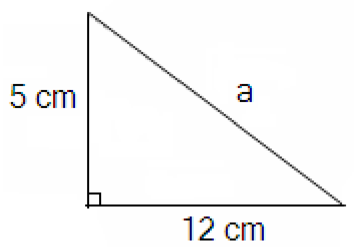


Applying the Pythagoras theorem

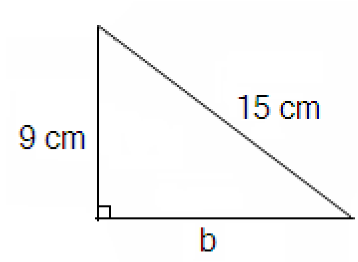
Example 11.1

Find the lengths of each side of the right angled triangles labeled in letters.

(a)



(b)



Solutions

(a) Using Pythagoras' theorem,

$a^2 = 5^2 + 12^2$

$= 25 + 144 = 169$

$a = \sqrt{169} = 13$ cm

(b) Similarly,

$15^2 = 9^2 + b^2$

$225 = 81 + b^2$

$b^2 = 225 - 81 = 144$

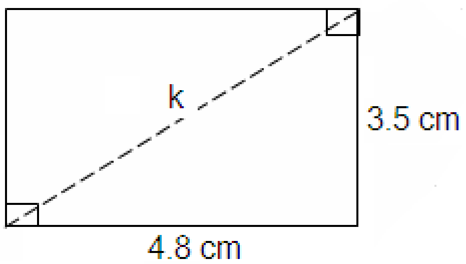
$b = \sqrt{144} = 12$ cm.

Example 11.2

If the lengths of the sides of a rectangle are 4.8 cm and 3.5 cm, find the length of its diagonal.

Solution

Let the length of the diagonal be k cm.



Then, $k^2 = 4.8^2 + 3.5^2$

$= 23.04 + 12.25$

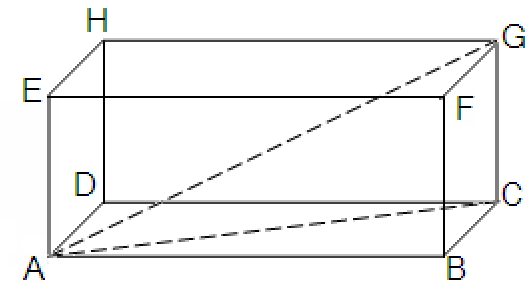
$= 35.29 \quad \therefore \quad k = \sqrt{35.29} = 5.940.$

Exercise 11.1

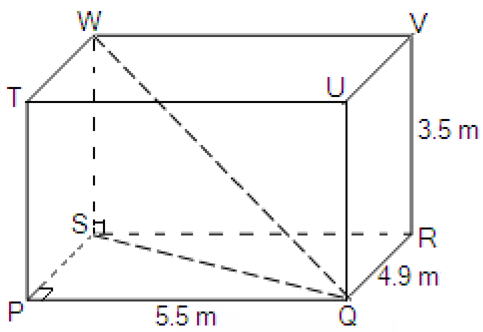
1. All the triangles in this question are right-angled. The three sides are x, y, z such that x is the hypotenuse.
   (a) y = 24 cm, z = 10 cm, calculate x.
   (b) x = 2.5 cm, z = 1.5 cm, calculate y.
   (c) y = 36 m, z = 15 m, calculate x.

(d) x = 37.08 cm, y = 23.26 cm, calculate z.

2. The sides of right-angled triangles are a, b and c, where c is the hypotenuse. Work out the following giving your answer correct to two decimal places.
   (a) a = 3 cm, b = 3 cm, calculate c.
   (b) a = 7 cm, b = 8 cm, calculate c.
   (c) a = 6.5 cm, c = 11.5 cm, calculate b.
   (d) c = 9 cm, b = 6 cm, calculate a.
   (e) c = 7.5 cm, a = 5 cm, calculate b.
   (f) c = 16 cm, b = 15 cm, calculate a.
3. Triangle ABC has AB = AC, BC = 10 cm and $\angle BAC = 90^0$. Calculate the lengths of sides AB and AC to the nearest millimeter.
4. The diagonal of a square is 15 cm. Calculate its perimeter.
5. The longest diagonal AG of a rectangular box is 12.5 cm. Diagonal AC of its base is 10 cm.

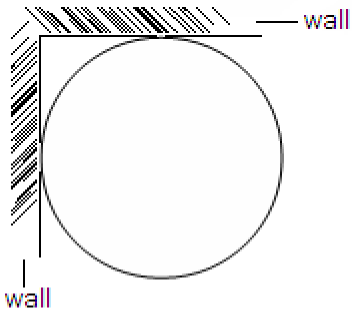


   Calculate the height of the box.
6. In a right angled triangle, the hypotenuse and one its shorter sides are 2.5 m and 1.5 m respectively. Determine the length of the third side.
7. A 2.6 m ladder has its foot 1.0 m from the base of a vertical wall. How far up the wall will it reach?
8. The length of a diagonal of a rectangular flower bed is 24.7 m and the length of one side is 19.8 m. Find the perimeter of the flower bed.
9. The area of a square field is 12,321 m^2. Find the length of the diagonal.
10. A room is in the shape of a cuboid.



If it is 5.5 m long, 4.9 m wide, 3.5 m high, find the distances from a corner of the floor to the opposite corner of the ceiling, WQ.

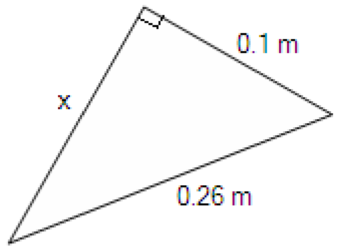
11. The figure drawn below shows a ball of radius 15 cm placed at a corner of a room.



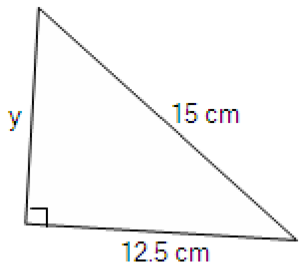
Find, to the nearest millimeter, the shortest distance from the corner of the room to the ball.

12. In each of the following figures, find the lengths of the sides marked in letters.

(a)

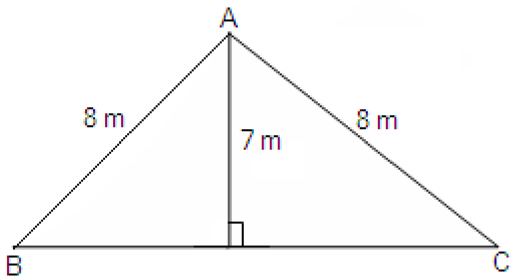


(b)



13. In triangle PQR, $\angle PRQ = 90^0$, PQ = 14 cm and PR = 8.9 cm. Calculate RQ.
14. A point P on the ground is 10 m from the base of a telephone pole. A wire from point P is attached to the pole at 12 m above the ground. Find the length of the wire.
15. A rectangular frame is made of metal bars. The length of its diagonal is 2.5 m. Find the length of the frame if the width of the frame is 1.5 m.

16. The diagram below shows part of a roof of a building.



If AB, BC, and AC are beams, find the length of BC.

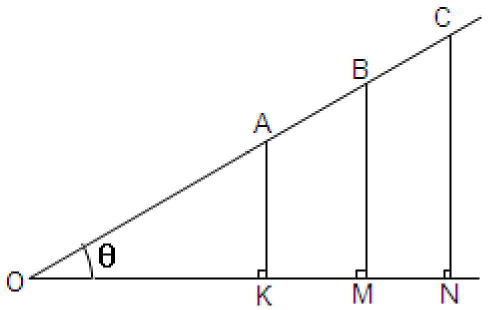
Chapter 12

THE TRIGONOMETRIC RATIOS

Trigonometric ratios express the relationship between the size of one angle and the lengths of two sides in a right-angled triangle.

The tangent of an angle

Draw an acute angle using a protractor. Take any three points on the slope and draw lines from them which are perpendicular to the horizontal line as shown in the figure below.

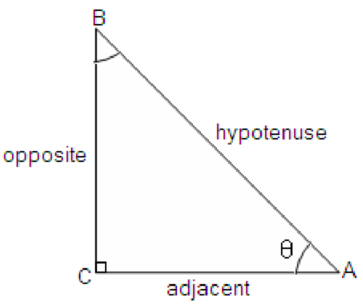


Point O is the intersection of the two lines which makes an angle θ, read as theta. Measure AK and OK. Determine the ratio $\frac{AK}{OK}$. Similarly, determine the ratios $\frac{BM}{OM}$ and $\frac{CN}{ON}$.

You will find that the values of the three ratios are approximately the same. This constant ratio for a given angle θ is called the tangent of θ and is denoted as tan θ.

Notice that triangles OAK, OBM and OCN whose sides have been used to find the ratios are right angled triangles. Our study of the tangent of an angle in this chapter will apply to right-angled triangles only.

Consider triangle ABC shown below. $\angle$BAC is θ. Side BC is **opposite** angle θ and AC is **adjacent** to angle θ.



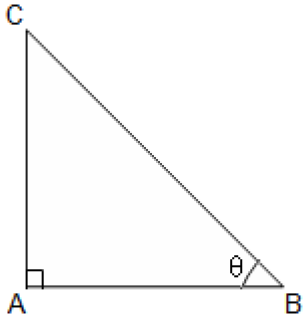
Side AB is the hypotenuse of the triangle. In a right-angled triangle, the tangent of an angle is the ratio of the opposite side to the adjacent side.

Thus, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

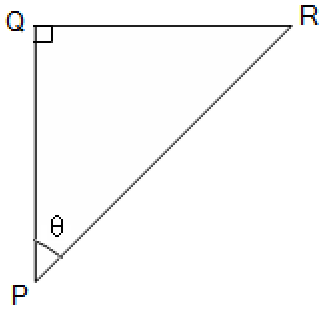
Exercise 12.1

1. Name the sides which are opposite and adjacent to the angle marked θ in each of the following triangles:

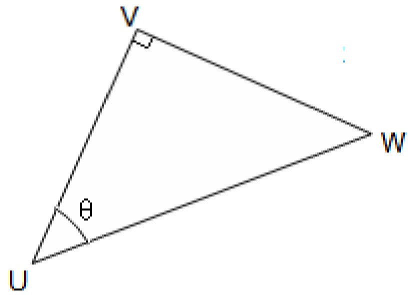
(a)



(b)

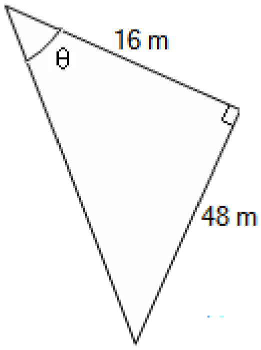


(c)

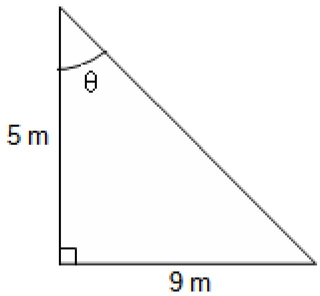


2. Express tan θ as a fraction in each of the following triangles:

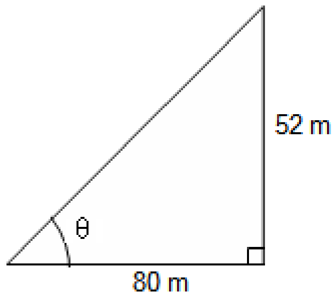
(a)



(b)



(c)



The table of tangents

When the angles are known, we use the table of tangents to find the ratios of the opposite to the adjacent sides. Angles in the table of tangents are expressed in degrees and points of degrees. They are also given in minutes. There are 60 minutes (60') in 1 degree (1^0). Thus, 60' is equivalent to 1^0. There are three major groups of columns in the table of tangents as in table 12.1.

Table 12.1: ***An extract of the table of tangents***

θ	**0'** 0.0^0	**6'** 0.1^0	**12'** 0.2^0	**18'** 0.3^0	**24'** 0.4^0	**30'** 0.5^0	**36'** 0.6^0	**42'** 0.7^0	**48'** 0.8^0	**54'** 0.9^0	**Differences** 1'	2'	3'	4'	5'
0	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0.0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0.0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0.0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0.0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0.1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0.1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0.1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0.1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0.1944	1962	1980	1998	2016	6035	2053	2071	2089	2107	3	6	9	12	15
12	0.2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0.2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0.2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0.2679	2698	2717	2736	2754	2773	2972	2811	2830	2849	3	6	9	13	16

1. The column headed θ gives angles in degrees.
2. The next 10 columns headed 0.0^0 to 0.9^0 or 0' to 54' give the tangents of angles. Angles increase in steps of 6' or one-tenth of a degree which is 0.1^0.
3. The last five columns headed 1' to 5' are called *differences* and indicate hundredths of degrees.

Example 12.1

Find tan 12.8^0.

Solution

To find tan 12.8^0, proceed as follows:
In the column headed θ, look for the row headed 12. Move along this row until you reach the column headed 0.8. The number at the intersection is 0.2272.
Therefore, tan 12.8^0 = 0.2272.

Note: 12.8^0 = $12^0$48'

In order to find tan $12^0$48' read the number where the row headed 12^0 meets with the column headed 48'.

Example 12.2

Find tan $12^0$45'

Tan $12^0$45' cannot be read directly from the table because there is no column for 45'. $12^0$45' lies between $12^0$42' and $12^0$48' both of whose tangents can be directly read from the tables.
Since $12^0$45' is 3' more than $12^0$42', the difference between their tangents is the number at the intersection of the row headed 12^0 and the column headed 3', that is, 0.0009. This difference is then added to the value of tan $12^0$42'. Thus, tan$12^0$45' = tan ($12^0$42' + 3')

= 0.2254 + 0.0009
= 0.2263

Note: $12^0 45'$ is also 3' less than $12^0 48'$.

The corresponding difference in their tangents is also 0.0009. This difference is subtracted from the value of tan $12^0 48'$.

Thus, tan$12^0 45'$ = tan $12^0 48'$ – tan 3'
= 0.2272 – 0.0009
= 0.2263

Exercise 12.2

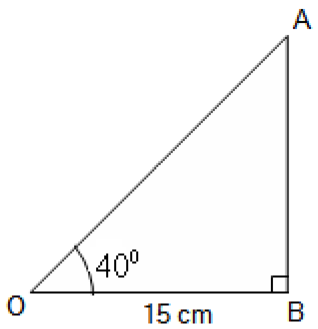
Use tables to find the tangents of the following angles:

1.	18^0	**2.**	80^0
3.	37^0	**4.**	5^0
5.	15.3^0	**6.**	44.9^0
7.	54.2^0	**8.**	$7^0 15'$
9.	$63^0 50'$	**10.**	$72^0 35'$
11.	$88^0 48'$	**12.**	$20^0 58'$

Using tangents in calculations

Example 12.3

Find length AB in triangle OAB



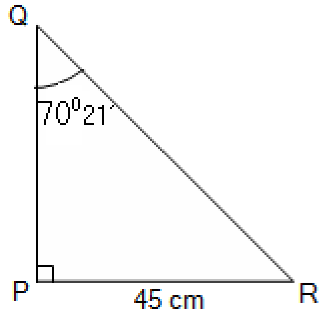
Solution

Tan 40^0 = $\frac{AB}{OB} = \frac{AB}{15}$

AB = 15 tan 40^0
= 15 × 0.8391 (from the tables)
= 12.6 cm.

Example 12.4

Find length PQ in triangle PQR.



Solution

Tan $70^0 21'$ = $\frac{45}{PQ}$

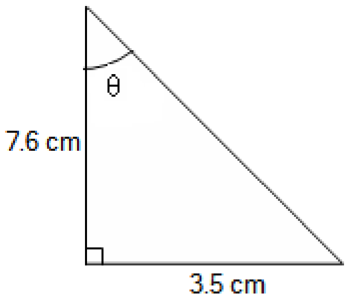
Hence, PQ = $\frac{45}{\tan 70^0 21'}$

= $\frac{45}{2.800}$

= 16.07 cm.

Example 12.5

Find the size of angle θ in the figure below.



Solution

Tan θ = $\frac{3.5}{7.6}$ = 0.4605

We need to find an angle, θ, whose tangent is 0.4605. In this case, the process of finding the tangent of an angle is reversed. From the table of tangents, locate the number 0.4605 or if it is missing, locate the nearest number smaller than it. The closest smaller tangent to 0.4605 is 0.4599, the tangent of $24^0 42'$. The difference between 0.4605 and 0.4599 is 0.0006. Look for 6 (or the closest number) in the difference columns along the row headed 24. The number closest to 6 is 7 in the column headed 2'. Thus, the angle whose tangent is 0.4605, is 2' more than $24^0 42'$.

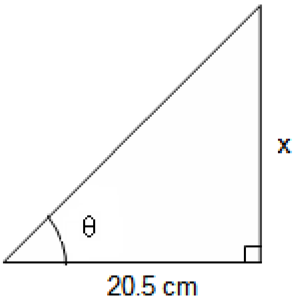
Therefore, θ = $24^0 42'$ + 2'

= $24^0 44'$

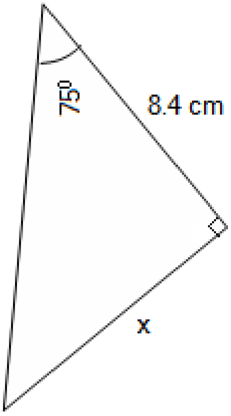
Exercise 12.3

1. Use tangents to find the lengths of the sides marked x in each of the following right-angled triangles:

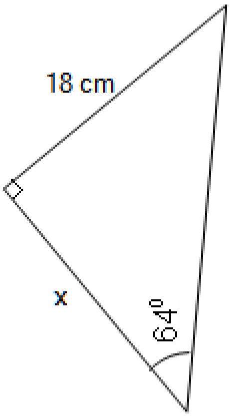
(a)



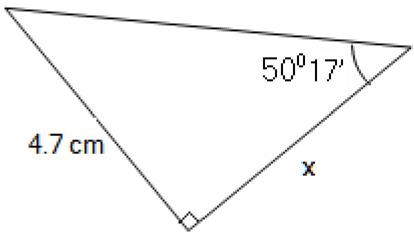
(b)



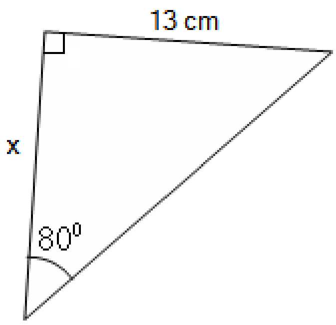
(c)



(d)

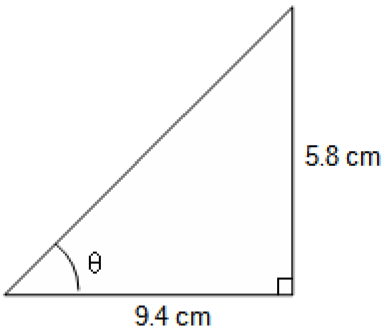


(e)

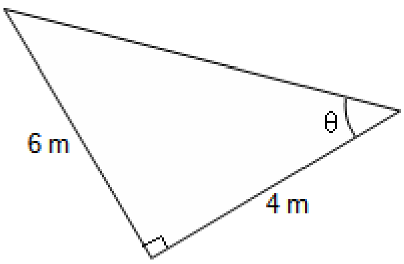


2. Use tangents to find the values of θ in each of the following right-angled triangles:

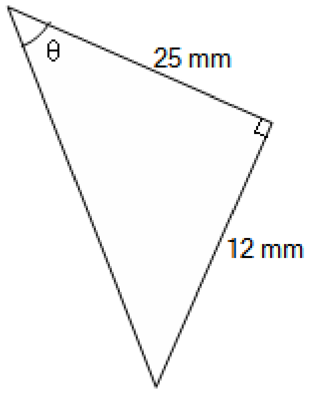
(a)



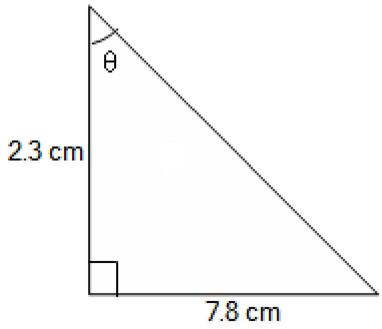
(b)



(c)



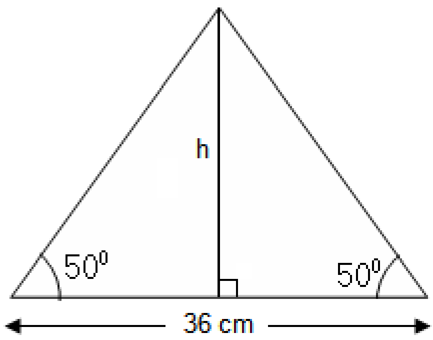
(d)



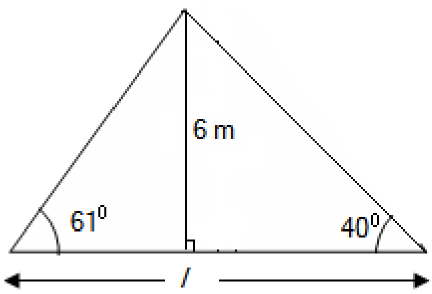
3. In $\triangle ABC$, $\angle ABC = 90^0$. Find:
   (a) AB, if BC = 182 cm and $\angle ACB = 33^0 40'$,
   (b) BC, if AB = 12 cm and $\angle BAC = 68^0 26'$,
   (c) $\angle BAC$, if BA = 26 mm and BC = 13 mm.

4. Find the values of the length marked in letters.

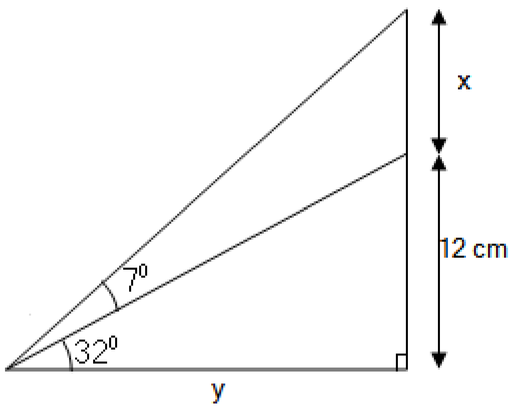
(a)



(b)



(c)



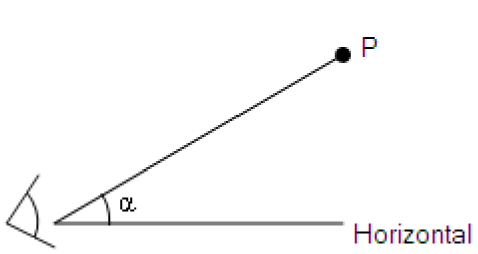
5. Calculate the acute angle between the diagonals of a rectangle that measures 20 cm by 14 cm.

6. An isosceles triangle has a base 80 cm long. Its height is 13 cm. Find the size of its angles.

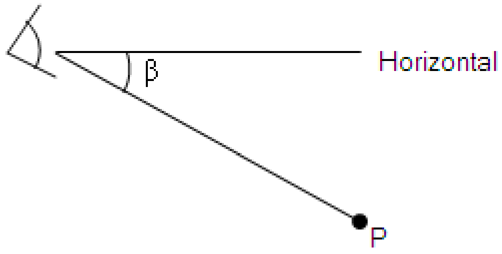
Applications of tangents

Elevation and Depression

Angles of elevation and depression give the direction of one point with respect to another in a vertical plane.

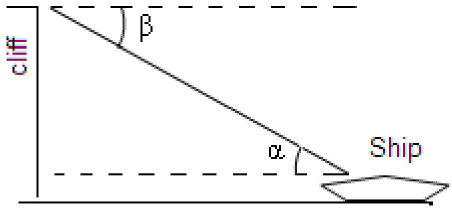


Used when looking up at a point, P



Used when looking down on a point, P

Angles of elevation and depression are always taken with the horizontal.



It can be seen that the angle of elevation from the ship to the top of the cliff is equal to the angle of depression from the top of the cliff to the ship (alternate angles).

Example 12.6

A boy, 120 cm tall, is standing 50 m from a flag post on a level ground. He finds that the angle of elevation to the top of the flag post is 15^0. Calculate the height of the flag post.

Solution

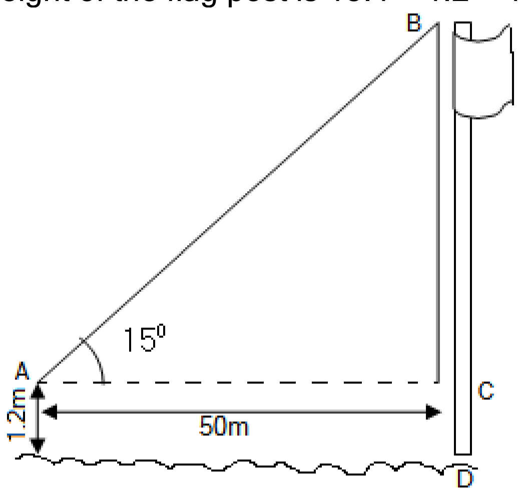
Tan $15^0 = \frac{BC}{50}$ (see diagram below)

Therefore, BC = 50 tan 15^0

= 50 × 0.2679

= 13.4 m.

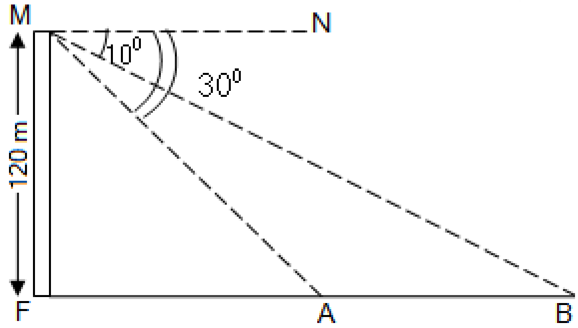
This is the height of the flag post above the boy's height of 1.2 m. Therefore, the height of the flag post is 13.4 + 1.2 = 14.6 m.



Example 12.7

A girl lying at the top of a cliff, 120 m high sees two rocks whose angles of depression are 10^0 and 30^0. If the rocks are in line with the foot of the cliff, find the distance between the rocks.

Solution



F is the foot of the cliff. A and B are positions of the rocks. We notice that $\angle NMB = \angle MBF$ and $\angle NMA = \angle MAF$ since they are alternate angles.

In ΔMFA, $\tan 30^0 = \frac{120}{FA}$

Therefore, $FA = \frac{120}{\tan 30^0} = \frac{120}{0.5774} = 208$ m.

In ΔMBF, $\tan 10^0 = \frac{120}{FB}$

Therefore, $FB = \frac{120}{\tan 10^0} = \frac{120}{0.1763} = 681$ m.

The distance between the rocks is

$AB = FB - FA = 681 - 208$ m $= 473$ m.

Alternatively, in ΔMFA, $\angle AMF = 60^0$.

Thus $\tan 60^0 = \frac{AF}{MF} = \frac{AF}{120}$

Therefore, AF = 120 × 1.132 = 208 m.

And in ΔFMB, $\angle BMF = 80^0$. Thus, $\tan 80^0 = \frac{FB}{MF}$

Therefore, FB = MF tan 80^0 = 120 × 5.671 = 681 m.

AB = FB - AF = 681 – 208 = 473 m.

Exercise 12.4

1. An angle of elevation of P from Q is 62^0. What is the angle of depression of Q from P?

2. What is the angle of elevation of the top of a building, 30 m tall, from a point on the ground 80 m away?

3. A ladder leans against a vertical wall making an angle of 15^0 with the wall. If its foot is 4 m from the wall, calculate the height above the ground of the top of the ladder.

4. A vertical tower, 40 m tall, casts a shadow on a level ground. Calculate the angle the rays of the sun make with the ground.

5. The angle of elevation to the top of a tree from a point on the ground 70 m from its foot is 18^0. Calculate the height of the tree.

6. Calculate the angles of a rhombus whose diagonals are 14 cm and 8 cm.

7. P is 15 km north of Q and R is 26 km west of P. Calculate the bearing of R from Q.

8. From the top of a rock, a boy sees a pawpaw tree 30 m away. He measures the angle of elevation of the top of the tree as 15^0 and the angle of depression at the bottom as 4^0. Calculate the height of the pawpaw tree.

9. Two boys standing on opposite sides of a tree 24 m tall, measure the angles of elevation of its top as 36^0 and 23^0. Calculate the distance between the two boys.

10. A tree that is 20 m from a busy path is being cut down. The angle of elevation of the top of the tree from the middle of the path is 50^0. Is it safe for the tree to fall in the direction of the path?

11. How far from the base of a 25 m cliff must a ship be so the angle of elevation of the top of the cliff from the ship is to be 7^0?

12. The angle of elevation of the top of a building is 70^0 from a point on the ground 50 m away.
    (a) How high is the building?
    (b) What would be the angle of elevation of the top of the building from a point on the ground 25 m away?

13. The angle of depression of a crossroads from an aero plane flying at 400 m is 20^0.
    (a) What is the horizontal distance of the plane from the crossroads?
    (b) What would the angle of depression have been if the plane had been flying at 600 m at the same horizontal distance away?

14. The angle of elevation of the top of a chimney on a house from a point on the ground 50 m away is 22^0.
    (a) How high above the ground is the top of the chimney?
    (b) If the angle of elevation of the roof of the house from the same point is 20^0, how tall is the house?
    (c) How tall is the chimney?

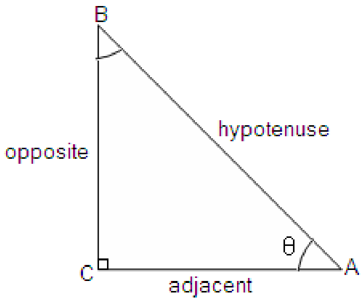
15. The angle of depression of a house from the top of a 175 m hill is 75^0.
    (a) How far is the hill from the house, horizontally?
    (b) If the angle of depression of a church from the top of the hill is 62^0, How far is the hill from the church, horizontally?
    (c) Assuming that the hill, house and church are all in the same straight line, how far is it from the house to the church?

16. (a) The angle of elevation of the top of a cliff from a ship at sea level is 12.3^0. If the ship is 2.3 km out to sea find the height of the cliff.

(b) A 50 m lighthouse stands on the top of the cliff. Find the angle of depression of the ship from the top of this lighthouse.

The sine of an angle

We have seen that for any right-angled triangle ABC, the ratio of the opposite side to the adjacent side is a constant fro a given angle, θ.



Similarly, the ratio of the opposite to the hypotenuse is a constant for any given angle, θ. This constant is called the **sine of angle θ**, abbreviated as sin θ and read as sine theta.

Thus, **sin θ** = $\frac{\textbf{opposite}}{\textbf{hypotenuse}}$

The table of sine

The table of sines is read in the same way as the table of tangents. For example, sin 24^0 = 0.4067 and sin $43^0 46'$ = 0.6909 + 0.0009 = 0.6918

Exercise 12.5

1. Write down the sines of each of the following angles:

(a)	13^0	(b)	76^0
(c)	44.3^0	(d)	$50^0 24'$
(e)	$32^0 19'$	(f)	$18^0 16'$

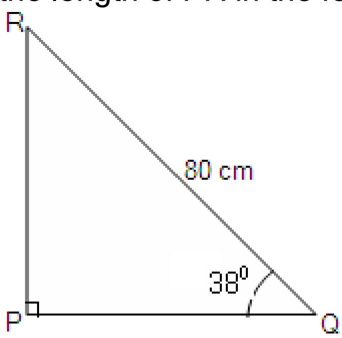
2. Write down the angles whose sines are:

(a)	0.8746	(b)	0.3272
(c)	0.8398	(d)	0.3555
(e)	0.6000	(f)	0.9927

Applications of sines

Example 12.8

Find the length of PR in the following diagram.



Solution

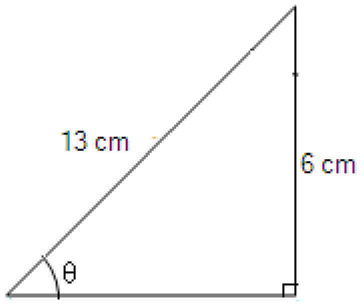
Sin $38^0 = \frac{PR}{RQ} = \frac{PR}{80}$

Thus, PR = 80 sin 38^0

= 80 × 0.6157 = 49.3 cm.

Example 12.9

Find the size of angle θ in the following diagram.



Solution

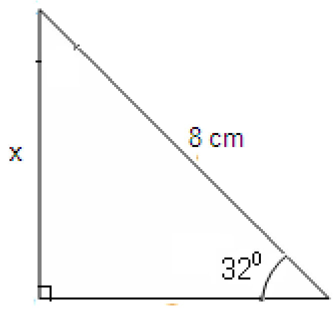
Sin θ = $\frac{6}{13}$ = 0.4615

θ = $27^0 29'$

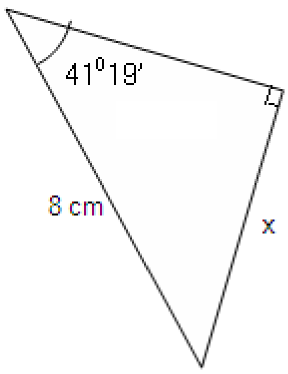
Exercise 12.6

1. Find the length of the side marked x in each of the following right-angled triangles:

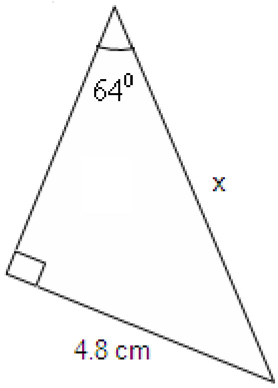
(a)



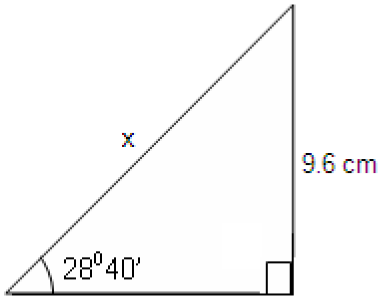
(b)



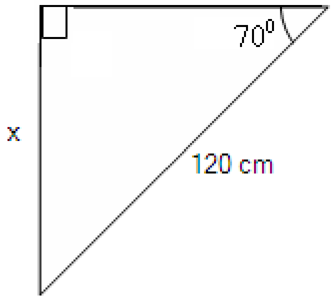
(c)



(d)

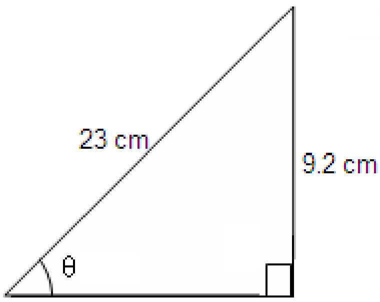


(e)

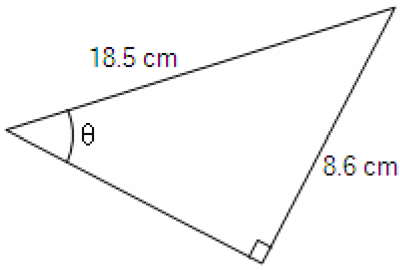


2. Find the size of the angle marked θ in each of the following right-angled triangles:

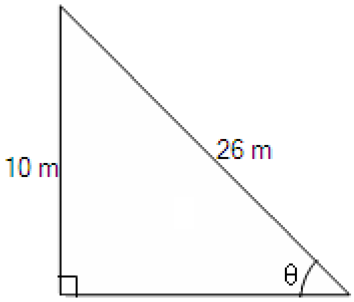
(a)

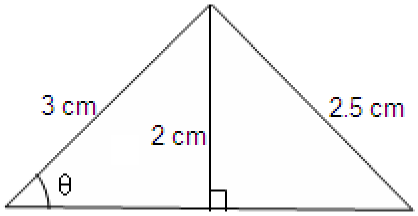


(b)

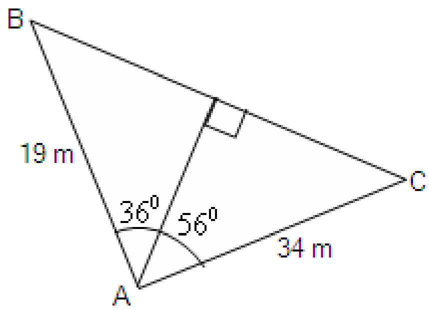


(c) (d)



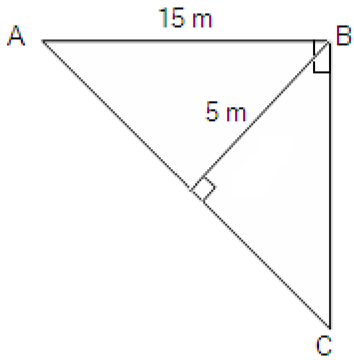


3. In triangle ABC below, find the length of BC.



4. An isosceles triangle with its base 18 cm long, has each of the identical angles as 43^0. Find the lengths of the identical sides.

5. In the figure below, triangles ABC and ABD are right-angled.



Find: (a) $\angle$BAD, (b) the length of BC.

6. Find the base length of an isosceles triangle whose vertex angle is 50^0 and the equal sides are 10 cm long.

7. An isosceles triangle has a base of 6 cm and the equal sides are each 8 cm long. Find the angles of the triangle and its height.

8. Find the angle subtended at the centre of a circle of radius 5 cm by a chord of length 3 cm.

The cosine of an angle

In any right-angled triangle, the ratio of the adjacent side to the hypotenuse is a constant for any given angle, θ. This constant ratio is called the **cosine of angle θ**, abbreviated as **cos θ**.

Thus, **cos θ** = $\frac{\text{adjacent}}{\text{hypotenus}}$.

The table of cosines

You will have noticed that the sines and tangents of angles increase as the angles increase. In the table of cosines, the cosine of angles **decrease** as the angles increase.

The cosines of angles are read in the same way as the sines except that the values in the columns of differences of the cosines are **subtracted** instead of being added.

Example 12.10

(a) Find cos $35^0 22'$.
(b) Find the angle whose cosine is 0.6890

Solutions

(a) From the tables, cos $35^0 22'$ can be found by getting cos $35^0 18'$
Cos $35^0 18' = 0.8161$
$35^0 22' - 35^0 18' = 4'$
Therefore, cos $35^0 22' = 0.816 - 0.0007 = 0.8154$

(b) From the tables, 0.6890 lies between 0.6664 and 0.6896 in the row headed 46. 0.6896 is 0.0006 more than 0.6890. We look for 6 or a number nearest to 6 in the differences column on the row headed 46^0. The number 6 is in the column 3'. The angle whose cosine is 0.6890 is $46^0 24' + 3' = 46^0 27'$.

Exercise 12.7

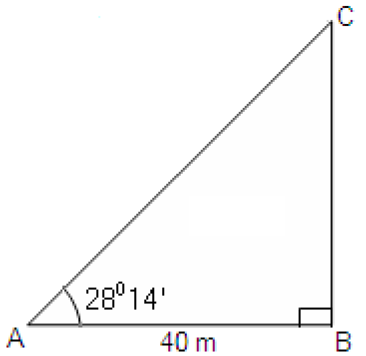
1. Find the cosine of each of the following angles:
(a) 23^0 (b) $48^0 17'$
(c) $67^0 9'$ (d) $83^0 34'$
(e) 42.6^0 (f) 15.7^0

2. Use tables to find the angles whose cosine are:
(a) 0.2554 (b) 0.0870
(c) 0.8895 (d) 0.5978
(e) 0.4970 (f) 0.9998

Applications of cosines

Example 12.11

Find AC in the following diagram.



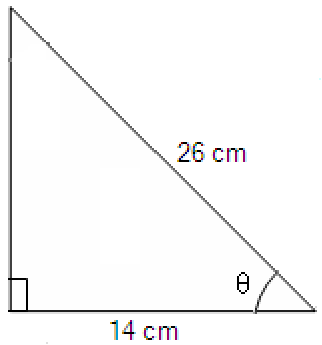
Solution

Cos $28^0 14' = \frac{AB}{AC} = \frac{40}{AC}$

$$AC = \frac{40}{\cos 28^0 14'} = \frac{40}{0.8810} = 45.4 \text{ m.}$$

Example 12.12

Find the size of angle θ in the following diagram.



Solution

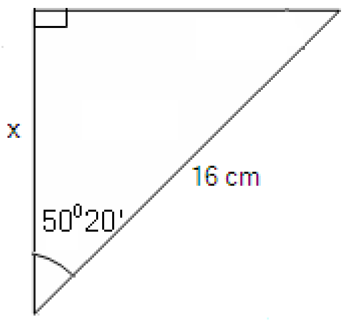
Cos $\theta = \frac{14}{26} = 0.5385$

$\theta = 57^0 25'$ (from tables)

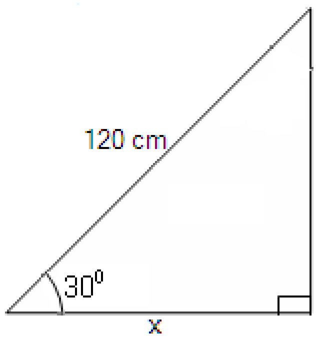
Exercise 12.8

1. Find the lengths of the sides marked x in the following right-angled triangles:

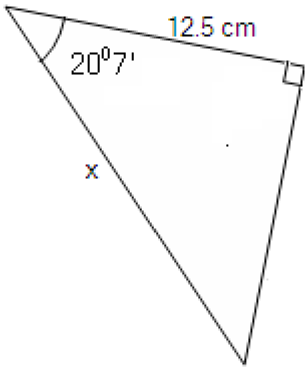
(a)



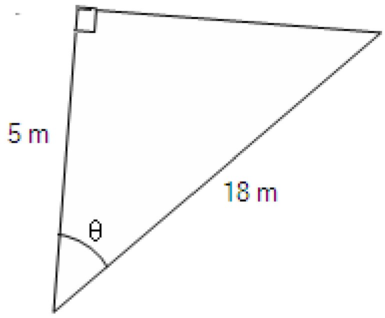
(b)

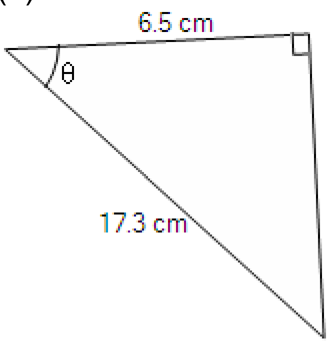


(c)



2. Find the value of θ in each of the following right-angled triangles:
(a) (b)





3. A diagonal of a rectangle is 20 m long and makes an angle of 40^0 with one of the sides. Calculate the lengths of the sides of the rectangle.

4. A rectangle 12 cm wide has a diagonal of 20 cm. Calculate the acute angle between the diagonals.

5. A ladder 8 m long is leaning against a vertical wall, with its foot 2 m from the wall. Calculate the angle the ladder makes with the floor.

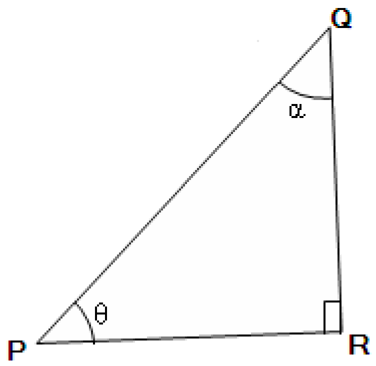
6. A chord of a circle 13 cm long subtends an angle of 118^0 at the centre. Find the radius of the circle.

7. A girl starts from village P, walks 300 m on a bearing of 038^0 to village Q, and then walks 60 m on a bearing of 073^0 to R. Find how far east R is from P.

8. An isosceles triangle ABC is such that AB = BC = 16.5 cm, and ABC = $80^0 24'$. Find the length of its altitude.

Tangents, sines and cosines

Consider a right-angled triangle PQR.



In the diagram above, $\sin\theta = \frac{QR}{PQ}$ and $\cos\theta = \frac{PR}{PQ}$.

Therefore, $\frac{\sin\theta}{\cos\theta} = \frac{QR}{PQ} \div \frac{PR}{PQ} = \frac{QR}{PQ} \times \frac{PQ}{PR} = \frac{QR}{PR}$

Also, $\tan\theta = \frac{QR}{PR}$.

Therefore, $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Example 12.14

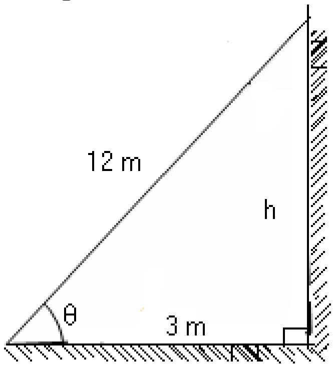
A metallic pipe 12 m long is leaning against a vertical wall, with its foot 3 m from the wall.

(a) Find the angle the pipe makes with the horizontal.

(b) Find the height of the wall where the pipe reaches.

Solution

Let the angle be θ and the height be h.



(a) $\cos\theta = \frac{3}{12} = 0.25$

Therefore, $\theta = 75^0 31'$

(b) $\sin 75^0 31' = \frac{h}{12}$

That is, $h = 12 \sin 75^0 31' = 12 \times 0.9682 = 11.6$ m.

The ratios of sine, cosine and tangent are best remembered by the acronym SOHCAHTOA.

Where, **SOH** means, **S**ine = $\frac{\text{Opposite}}{\text{Hypotenuse}}$

CAH means, **C**osine = $\frac{\text{Adjacent}}{\text{Hypotenuse}}$

TOA means. **T**angent = $\frac{\text{Opposite}}{\text{Adjacent}}$.

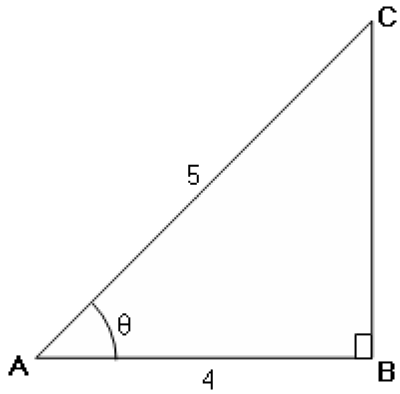
Example 12.15

Find without using tables, the value of sin θ and tan θ, if cos θ = $\frac{4}{5}$ and θ is an acute angle.

Solution

Cos θ = $\frac{\text{adjacent}}{\text{hypotenuse}}$.

Draw ΔABC and indicate the sides as in the following diagram.



AB = 4 units and AC = 5 units. Using Pythagoras' theorem,

$BC^2 = AC^2 - AB^2$

$= 5^2 - 4^2$

$= 9$

BC $= 3$

Sin θ = $\frac{BC}{AC} = \frac{3}{5}$ and tan θ = $\frac{BC}{AB} = \frac{3}{4}$

Also, tan θ = $\frac{\sin\theta}{\cos\theta}$ = $\frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4}$.

Exercise 12.9

1. Given that tan x = $\frac{3}{4}$, find cos x and sin x.

2. Given that cos x = $\frac{8}{17}$, find sin x and tan x.

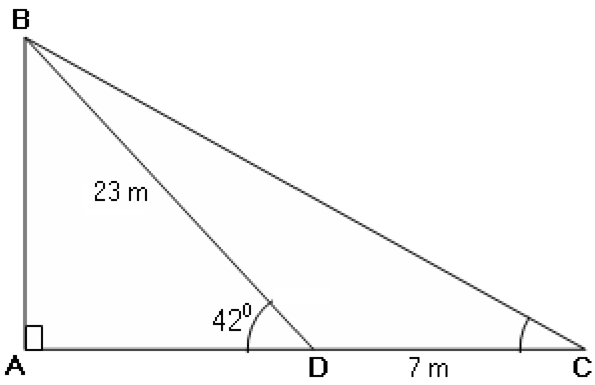
3. Given that sin x = $\frac{5}{13}$, find cos x and tan x.

4. Given that tan x = $\frac{12}{5}$, find cos x and sin x.

5. If tan x = $\frac{5}{12}$, evaluate $\frac{\cos x - \sin x}{\sin x + \cos x}$

6. The shadow of a building, on a horizontal ground, is twice as long as its height. If its height is 25 m, calculate the angle of elevation to the sun from the ground.

7. In the figure below, calculate the lengths of lines:
   (a) AD (b) AB
   (c) BC


B
23 m
42^0
A
D
7 m
C